An introduction to the eXtended Finite Element Method (X-FEM)

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  - Then Liège...
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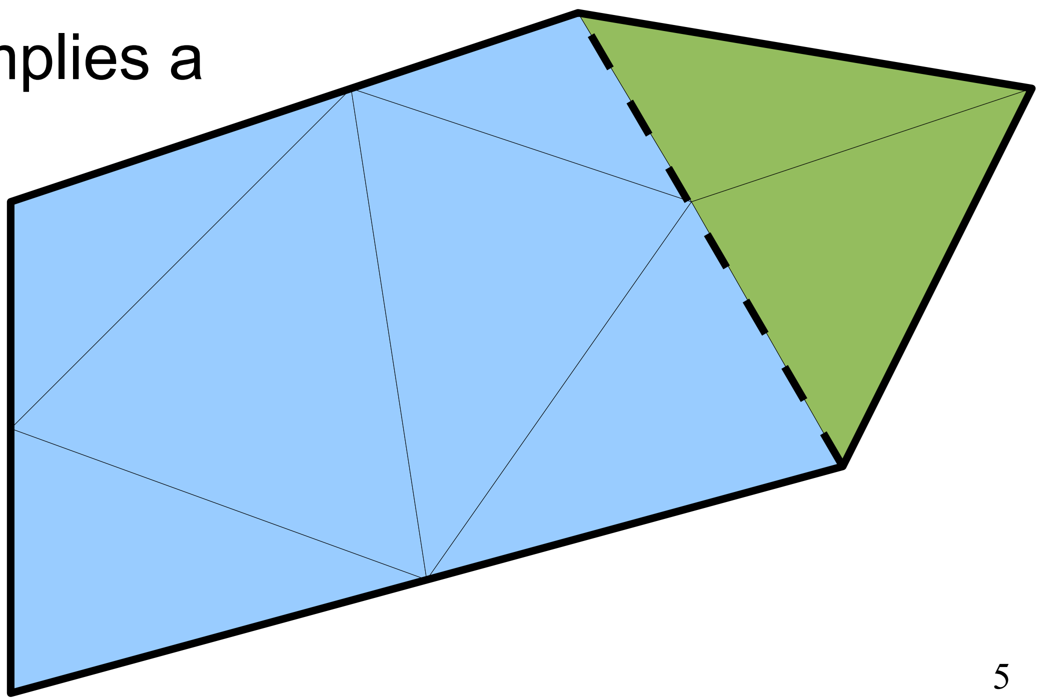
Lecture plan

- Introduction
- Reminder
- Simple problems (jump on the primal variable)
- Extensions in 2D / 3D
- Other types of problems (jump on the derivatives)
- Other applications
- Boundary conditions
- References

Course Notes available at :

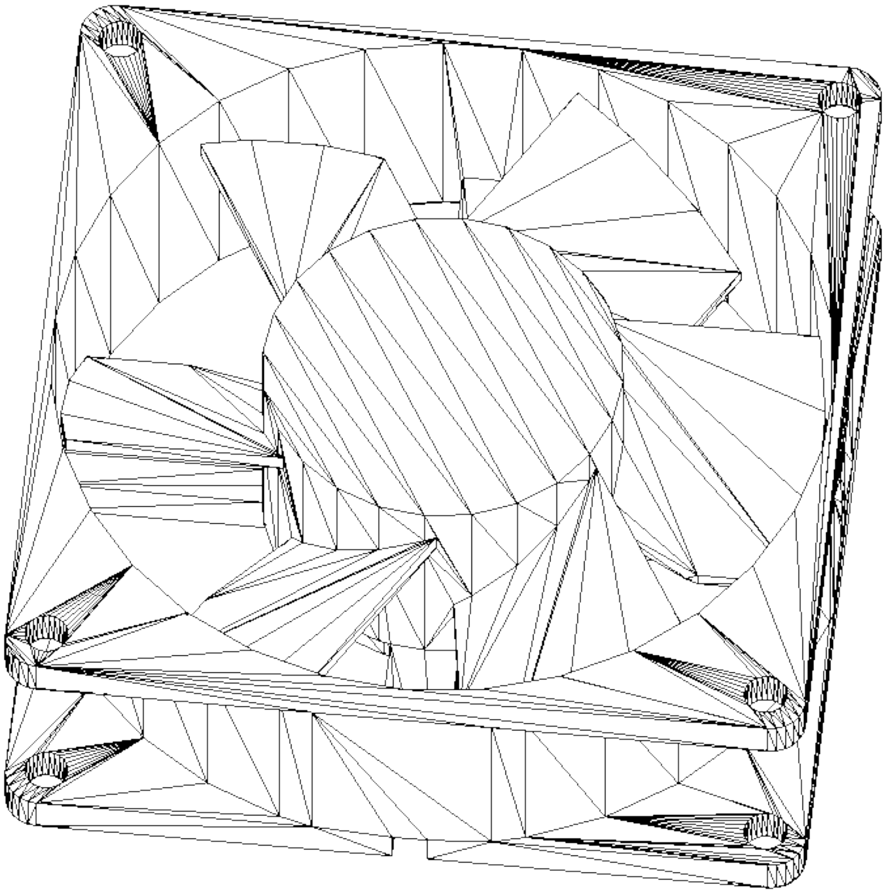
https://www.cgeo.uliege.be/X-FEM

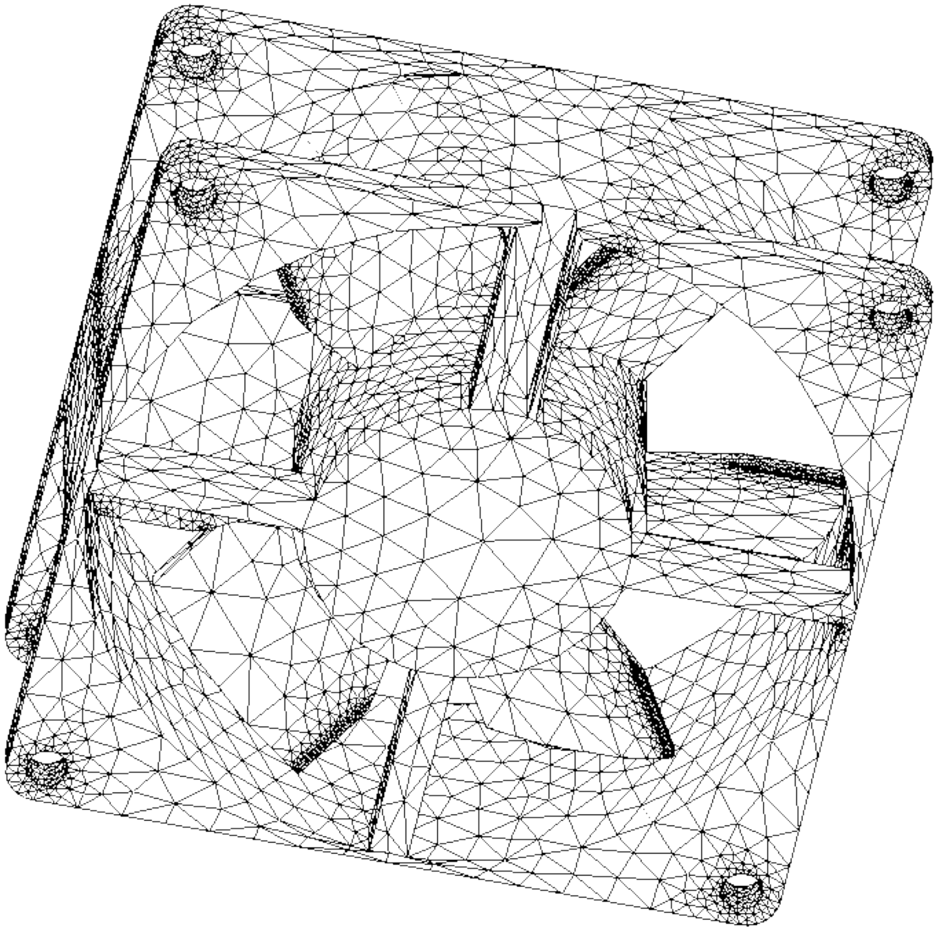
- "Classical" finite element computation
  - The geometry is bounded by element sides
    - Bounds the computation domain
    - Bounds the interface between zones of dissimilar properties
    - A change in geometry implies a change in the mesh
    - Time evolving problems may induce remeshing at each time step in the computation



- Mesh generation techniques
  - May be costlier than the sole finite element computation
  - (Often) necessitates a strong human interaction
  - Are a potential source of mistakes
    - Of human origin
    - Or from the lack of robustness of remeshing algorithms

Introduction

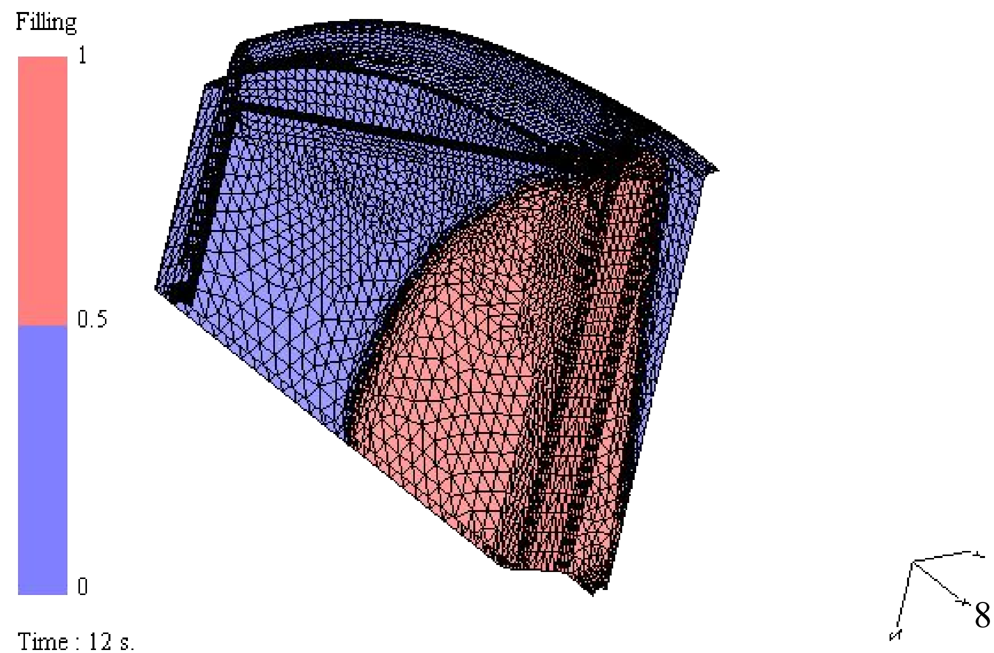




Introduction

- The idea here:
  - Minimize the constraints on the mesh that is used in the FEM simulations
  - However, mesh generations is still necessary
    - e.g. the accuracy of the computation depents on the quality of the mesh

      → mesh adaptation



Reminder

The method relies on the classical FEM; starting with the weak form of a physical problem :

$$\text{Find } u \in H^1(\Omega) \text{ such that}$$
$$\int_\Omega a(u,v)\, d\Omega = \int_\Omega b(v)\, d\Omega \quad \forall v \in H_0^1(\Omega)$$

Discretization: One looks for u in a discrete function space $V_h \subset H^1(\Omega)$ (trial functions v belong to the same space $V_{0h} \subset H_0^1(\Omega)$)

$$u_h(x) = \sum_i \lambda_i N_i(x) \quad , \; x \in \Omega$$

Reminder

- A space-conforming mesh is used to define the shape functions SFs $u(x)=\sum_k \lambda_k N_k$ for $x \in T_j$
- They have a compact support
- Partition of unity $\sum_i N_i = 1$
- Interpolation $u(x_i)=\lambda_i$

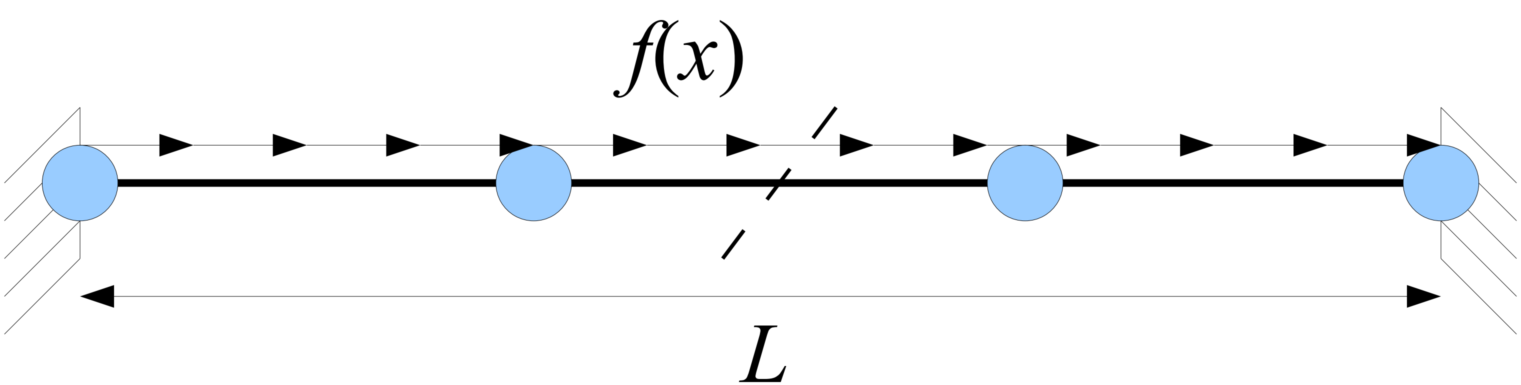
Reminder

- SFs with a compact support
  - Allows to have banded or hollow matrices (low memory imprint & good solver performance)
- Partition of unity
  - One is able to represent a constant field !
- Interpolation
  - Easy to impose Dirichlet boundary conditions
- Use of conforming meshes
  - Pre-computations of many operators is possible at an elementary level

Extended Finite Elements

Simple problem

- Clamped 1D rod (L, E, S) with a variable load $f(x)$
- One wants to get the displacement $u(x)$ and assume that the rod is cut at some place
  - With the classical FEM
  - With the eXtended Finite Element Method



Simple problem

- Weak form, with homog. boundary conditions

$$\text{find } u \in H_0^1(\Omega) \text{ such that}$$
$$a(u,v) = b(v)\ \forall v \in H_0^1(\Omega)$$

with

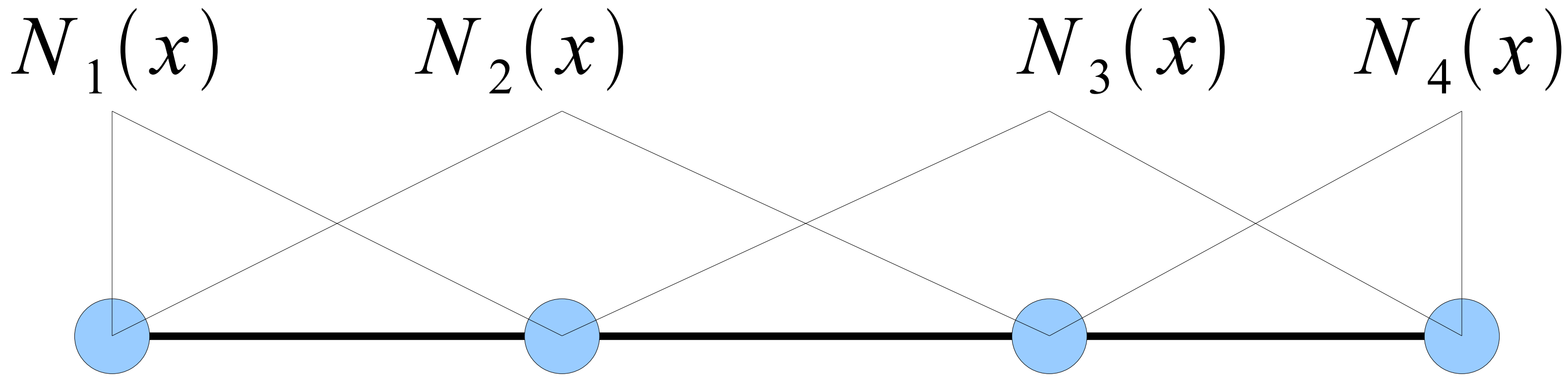
Elementary (stiffness) matrix

$$a(u,v) = \int_0^L ES\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\right)dx \qquad b(v) = \int_0^L f(v)dx$$

Elementary vector (loads)

Simple problem

- Discretization : Linear elements, nodal shape functions.



$$u_h(x) = \sum_i \lambda_i N_i(x)$$

Simple problem

- By reporting the discrete form of u and v in the weak form, one gets the following linear system :

$$\begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}$$

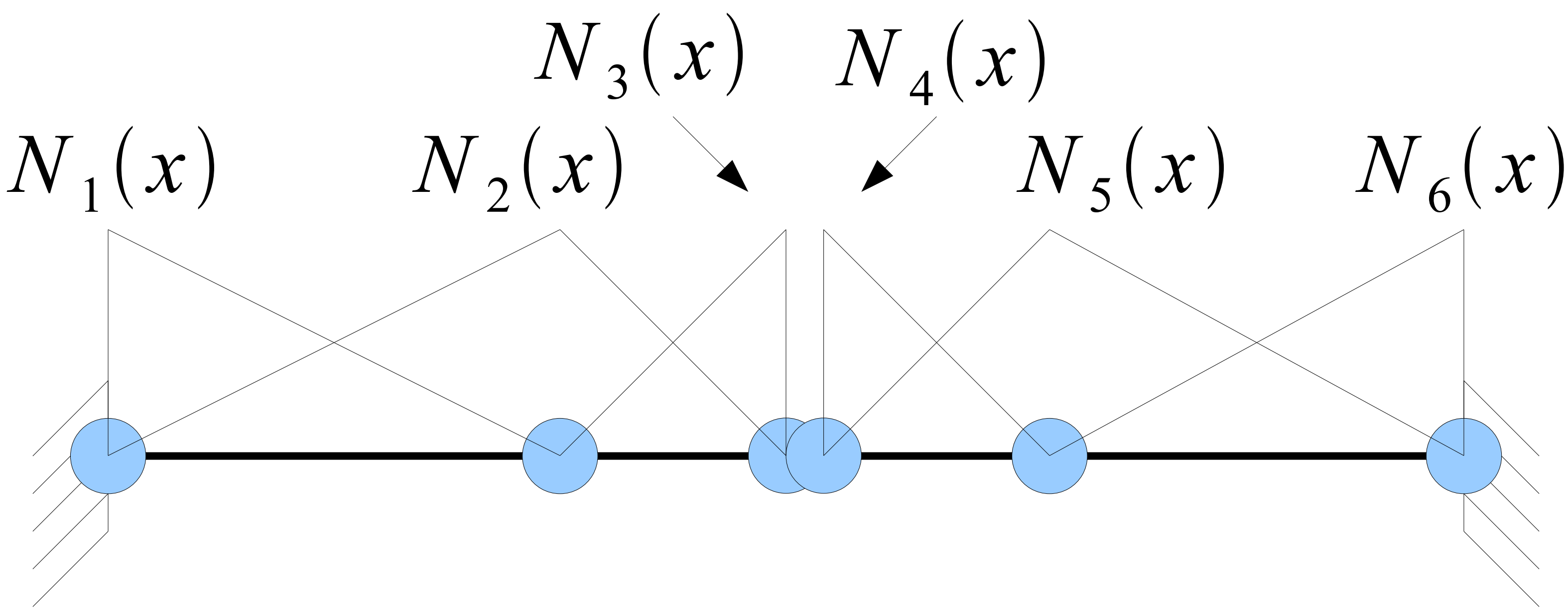
$$k_{ij} = \int_0^L ES \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} dx$$

$$f_i = \int_0^L N_i \cdot f(x)\, dx$$

- Here, coefficients λ_1 and λ_4 vanish (clamped extremities)

Cut the rod : FEM case

- Add two nodes and do the same
  - This is “remeshing”, it is simple, fast and robust in 1D, less 2D and much less in 3D



Cut the rod : FEM case

- After discretizing, one gets :

$$\begin{bmatrix} k_{22} & k_{23} & 0 & 0 \\ k_{32} & k_{33} & 0 & 0 \\ 0 & 0 & k_{44} & k_{45} \\ 0 & 0 & k_{54} & k_{55} \end{bmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix}$$

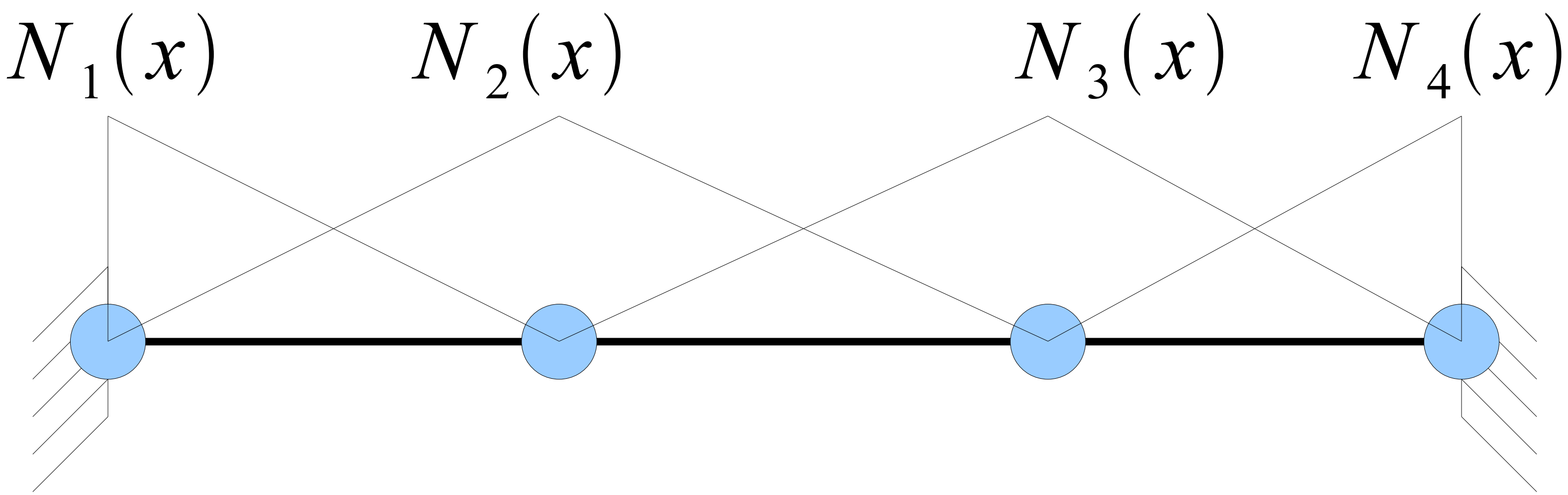
  - The two outlined parts are independent
  - One could solve the linear system separately for each sub-problem

Cut the rod : FEM case

- The meaning of the DoFs is kept (λ_i means the displacement of node i.)
- There is indeed a discontinuity in the displacement at nodes 3 and 4
- Nothing changes in the implementation – only the mesh and its topology are modified

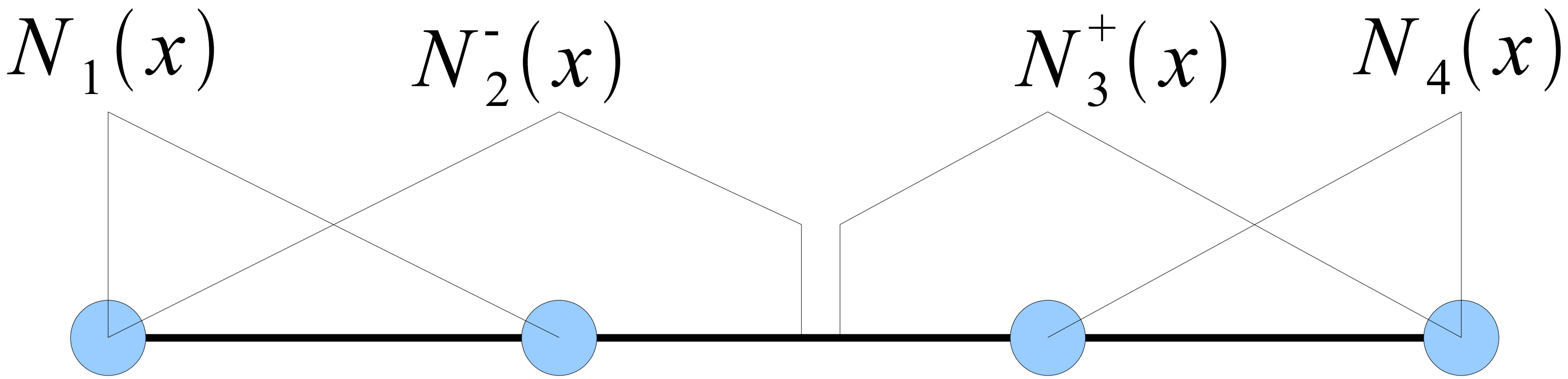
Cut the rod : X-FEM case

- Now : we don't change the mesh !
- But one can add/modify shape functions



Cut the rod : X-FEM case (I)

- Case (I) :



Cut the rod : X-FEM case (I)

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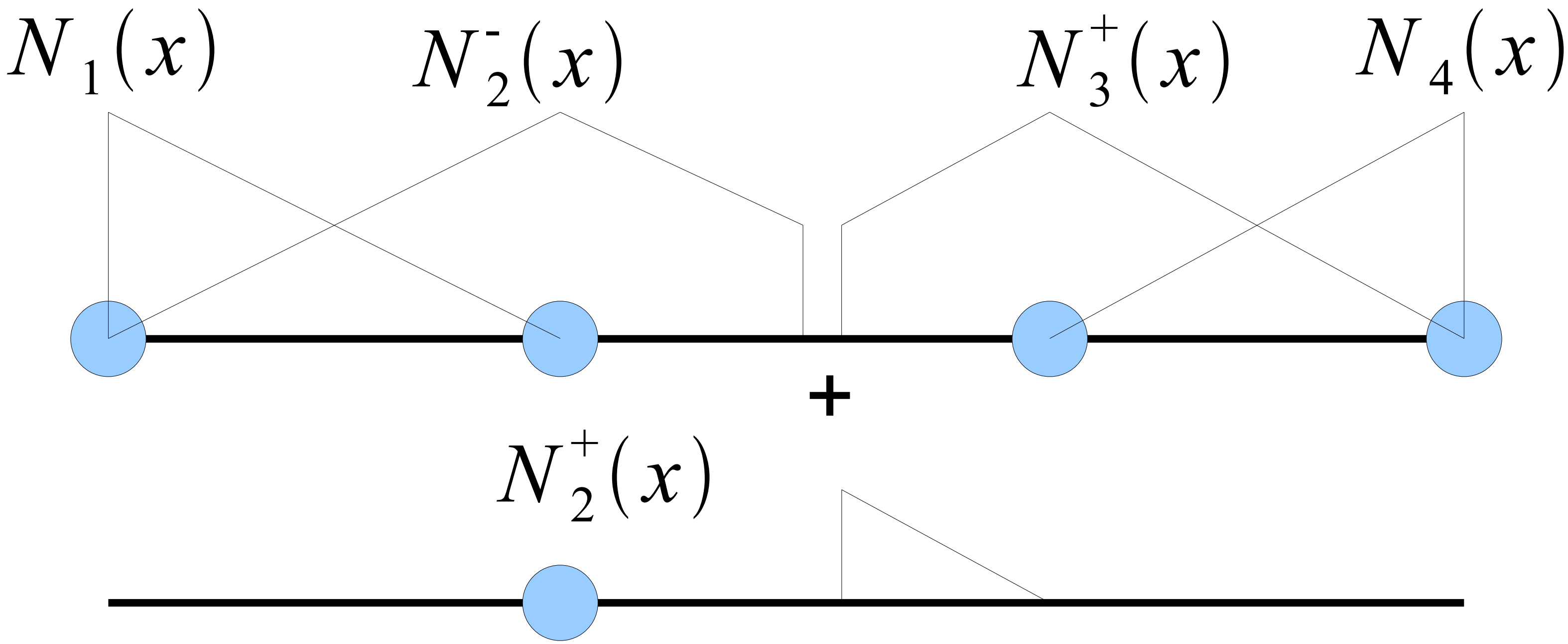
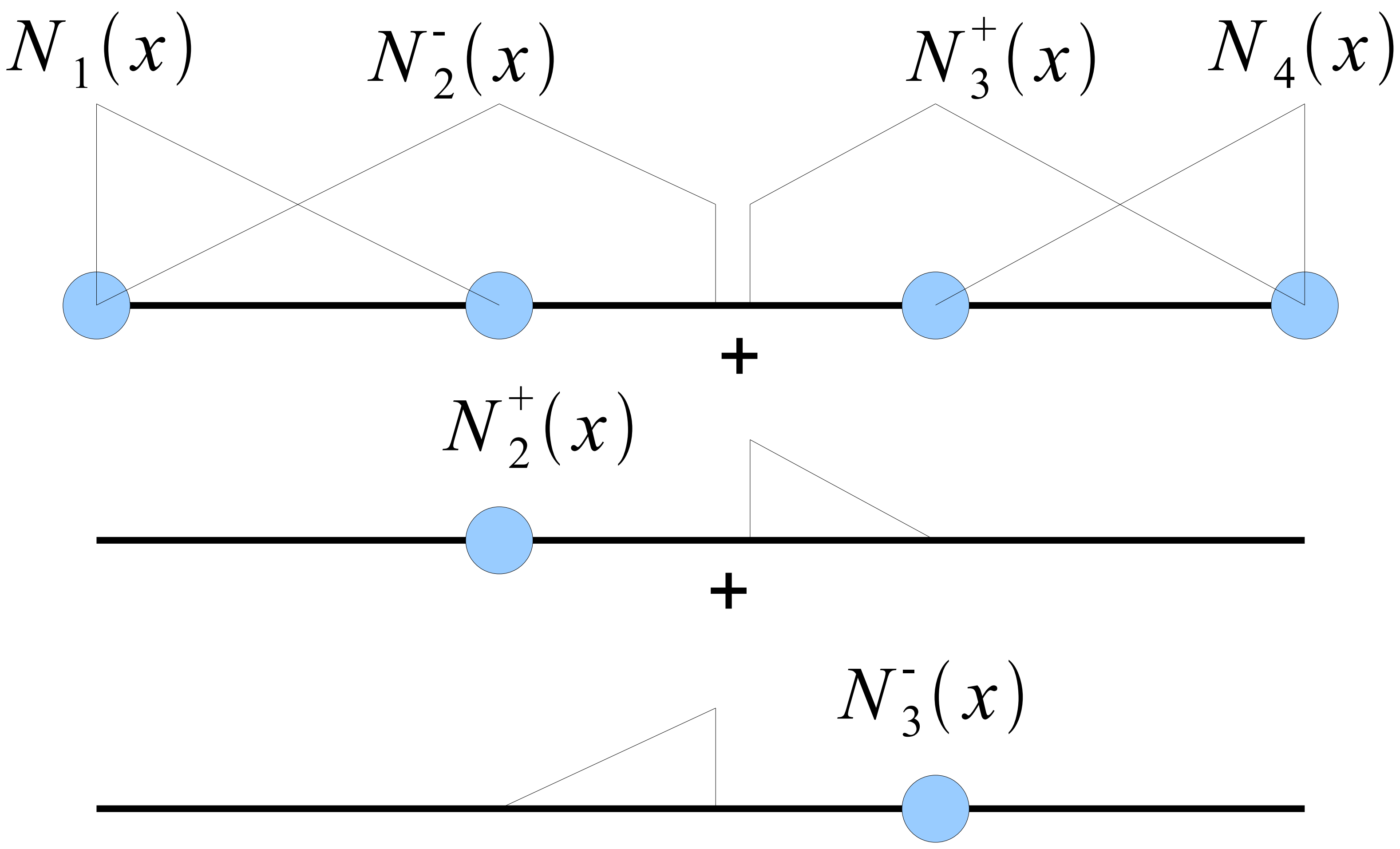


Cut the rod : X-FEM case (I)

- Case (I) :



Cut the rod : X-FEM case (I)

- How to compute the $N_j^{+,-}$ from the N_i ?
  - Let's introduce the Heaviside function :

$$H(s)=\begin{cases} 0 & \text{if} \quad s\leq 0 \\ 1 & \text{if} \quad s>0 \end{cases}$$

  - This is its complement :

$$\bar{H}(s)=\begin{cases} 1 & \text{if} \quad s\leq 0 \\ 0 & \text{if} \quad s>0 \end{cases}$$

  - s is the distance to the cut (here, $s=x-\dfrac{L}{2}$)

Cut the rod : X-FEM case (I)

- With these notations, one have :

$$\begin{cases} N_i^+(x) = N_i(x)\cdot H(s) \\ N_i^-(x) = N_i(x)\cdot \bar{H}(s) \end{cases}$$

- One may notice that the partition of unity is preserved

Cut the rod : X-FEM case (I)

- One has to sort out the mesh nodes
  - Those which have “regular” degrees of freedom go into set “N”
  - Those which have modified degrees of freedom go into set “C”
- The solution field u is written as :

$$u(x)=\sum_{i\in \mathrm{N}}\lambda_i N_i(x)+\sum_{j\in \mathrm{C}}\lambda_j^+ N_j^+(x)+\sum_{k\in \mathrm{C}}\lambda_k^- N_k^-(x)$$

Cut the rod : X-FEM case (I)

- Linear system
  - We number the DoFs as follows :

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \lambda_1 & \lambda_2^- & \lambda_3^- & \lambda_2^+ & \lambda_3^+ & \lambda_4 \end{bmatrix}$$

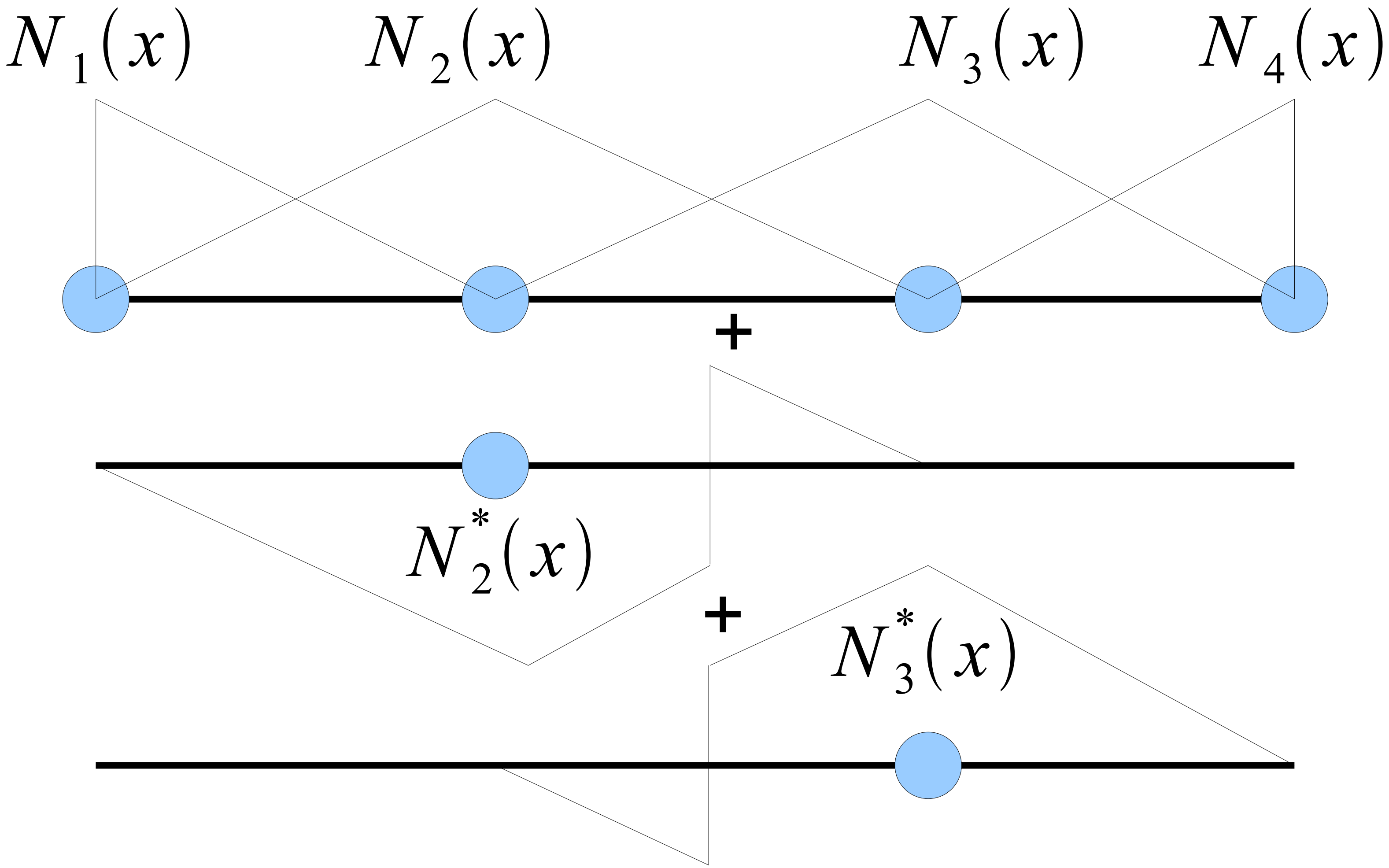
$$\begin{bmatrix} k_{22}^- & k_{23}^- & 0 & 0 \\ k_{32}^- & k_{33}^- & 0 & 0 \\ 0 & 0 & k_{22}^+ & k_{23}^+ \\ 0 & 0 & k_{32}^+ & k_{33}^+ \end{bmatrix} \cdot \begin{pmatrix} \lambda_2^- \\ \lambda_3^- \\ \lambda_2^+ \\ \lambda_3^+ \end{pmatrix} = \begin{pmatrix} f_2^- \\ f_3^- \\ f_2^+ \\ f_3^+ \end{pmatrix}$$

Cut the rod : X-FEM case (I)

- Again, we manage to separate the domain in two parts
- The signification of the degrees of freedom is partly lost
- Some shape functions have to be modified
- Two “Heaviside” functions are needed to modify the shape functions

Cut the rod : X-FEM case (II)

- Without changing the shape functions ! (case II)



Cut the rod : X-FEM case (II)

- How to compute the N_j^* from the N_i ?
  - Lets introduce the modified Heaviside function :

$$H^*(s) = 2H(s) - 1 = \begin{cases} -1 & \text{if } s \leq 0 \\ 1 & \text{if } s > 0 \end{cases}$$

- With this notation, one finds that :

$$N_i^*(x) = N_i(x) \cdot H^*(s)$$

Cut the rod : X-FEM case (II)

- One should again sort out the mesh nodes
  - Those which have modified DoFs go into set "C"
  - "regular" shape functions are still everywhere (no change with respect to regular FEM in that case)
- The solution field u is written as :

$$u(x) = \sum_{i \in \Omega} \lambda_i N_i(x) + \sum_{j \in \mathrm{C}} \lambda_j^* N_j^*(x)$$

Cut the rod : X-FEM case (II)

- Linear system
  - We number the DoFs as follows :

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \lambda_1 & \lambda_2 & \lambda_2^* & \lambda_3 & \lambda_3^* & \lambda_4 \end{bmatrix}$$

$$\begin{bmatrix} k_{22} & k_{22^*} & k_{23} & k_{23^*} \\ k_{2^*2} & k_{2^*2^*} & k_{2^*3} & k_{2^*3^*} \\ k_{32} & k_{32^*} & k_{33} & k_{33^*} \\ k_{3^*2} & k_{3^*2^*} & k_{3^*3} & k_{3^*3^*} \end{bmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_2^* \\ \lambda_3 \\ \lambda_3^* \end{pmatrix} = \begin{pmatrix} f_2 \\ f_2^* \\ f_3 \\ f_3^* \end{pmatrix}$$

Cut the rod: X-FEM case (II)

- At the matrix level, the two “parts” are linked
- Are there two physically separated domains ?
  - Lets assemble the matrix without taking care of the boundary conditions, and then determine the number of vanishing (singular) eigenvalues of this matrix.
    - If there is only one domain, there will be only one singular eigenvalue (corresponding to the missing Dirichlet BC to get a non singular system)
    - Two singular values → the rod is indeed cut in two, and two Dirichlet boundary conditions are needed.

Cut the rod : X-FEM case (II)

- Case without cut and without BC : typical matrix

$$K^s = k \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\det\left(K^s - \alpha\,\mathrm{I}\right) = 0$$

$$k = \frac{3ES}{L}$$

```
octave:27> K
K =

   1  -1   0   0
  -1   2  -1   0
   0  -1   2  -1
   0   0  -1   1

octave:28> E=eig(K)
E =

  -2.67429966923143e-17
   5.85786437626905e-01
   2.00000000000000e+00
   3.41421356237310e+00

octave:29>
```

One eigenvalue vanishes.

Cut the rod : X-FEM case (II)

- Case with a cut and without BC : typical matrix

$$K^c = k \cdot \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 1 & -1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\det\left(K^c - \alpha \mathrm{I}\right) = 0$$

```
octave:17> K
K =

   1  -1   1   0   0   0
  -1   2  -1  -1   0   0
   1  -1   2   0  -1   0
   0  -1   0   2   1  -1
   0   0  -1   1   2  -1
   0   0   0  -1  -1   1

octave:18> E=eig(K)
E =

  -1.30983399108489e-16
  -4.764701360007920e-17
   1.000000000000000e+00
   1.000000000000000e+00
   4.000000000000000e+00
   4.000000000000000e+00

octave:19>
```

Two eigenvalues vanished : it is OK !

Cut the rod : X-FEM case (II)

- The meaning of the degrees of freedom is lost
- One keeps classical FE basis functions and add others by enrichment
  - A kind of hierarchical FE basis is built
- Only one enrichment function (simpler !)

Cut the rod : X-FEM case

- Cases (I) and (II) are equivalent (the results are exactly identical)

  We indeed have a linear combination between shape functions of (I) and those of (II) :

$$N_2(x) = N_2^+(x) + N_2^-(x) \qquad N_3(x) = N_3^+(x) + N_3^-(x)$$
$$N_2^*(x) = N_2^+(x) - N_2^-(x) \qquad N_3^*(x) = N_3^+(x) - N_3^-(x)$$

- The case (II) is part of the more theoretical frame – use of a given enrichment function and “constructive” synthesis.

Definition

- eXtended Finite Element Method
  - It is based on classical FEM basis functions
  - The product between these functions and a given enrichment function $E_k(x)$ is then added
  - These enriched functions are able to represent a specific behavior of the solution field that classical shape functions are unable to represent efficiently. (e.g. a discontinuity)

$$u(x) = \sum_{i \in \Omega} \lambda_i N_i(x) + \sum_k \sum_{j \in C} \lambda^*_{jk} N_j(x) \cdot E_k(x)$$

Lecture plan

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- Extensions in 2D / 3D
- Other types of problems (jump on the derivatives)
- Other applications and current research
- Boundary conditions
- References

Extended Finite Elements

In 2D / 3D

- Case of linear elasticity
- Representation of cracks
- Level-sets
- Crack propagation

2D Example

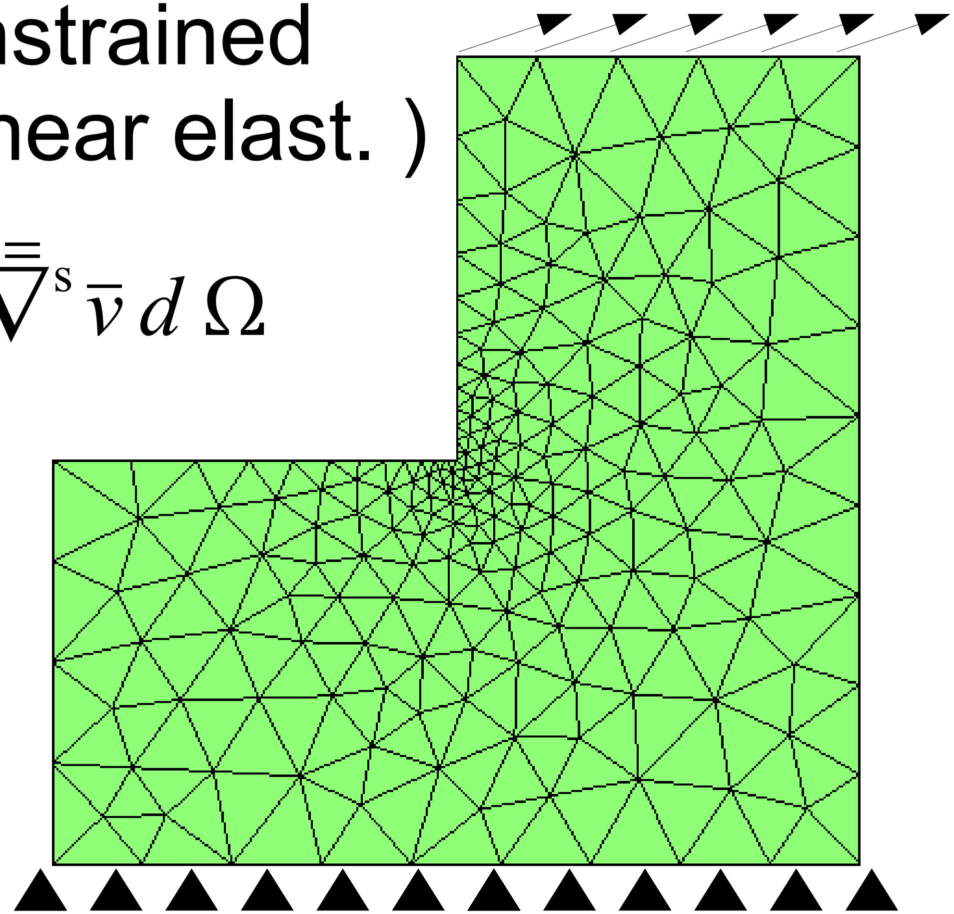
- A wedge with constrained displacements (linear elast.)

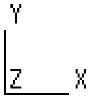
$$a(\bar{u},\bar{v})=\int_{\Omega} \bar{\bar{\nabla}}^{s}\bar{u}:\bar{\bar{\bar{\bar{D}}}}:\bar{\bar{\nabla}}^{s}\bar{v}\,d\Omega$$

$$b(\bar{v})=\int_{\Omega} \bar{f}\cdot\bar{v}\,d\Omega$$

find $\bar{u}$ such that

$$a(\bar{u},\bar{v})=b(\bar{v}) \quad \forall\,\bar{v}$$



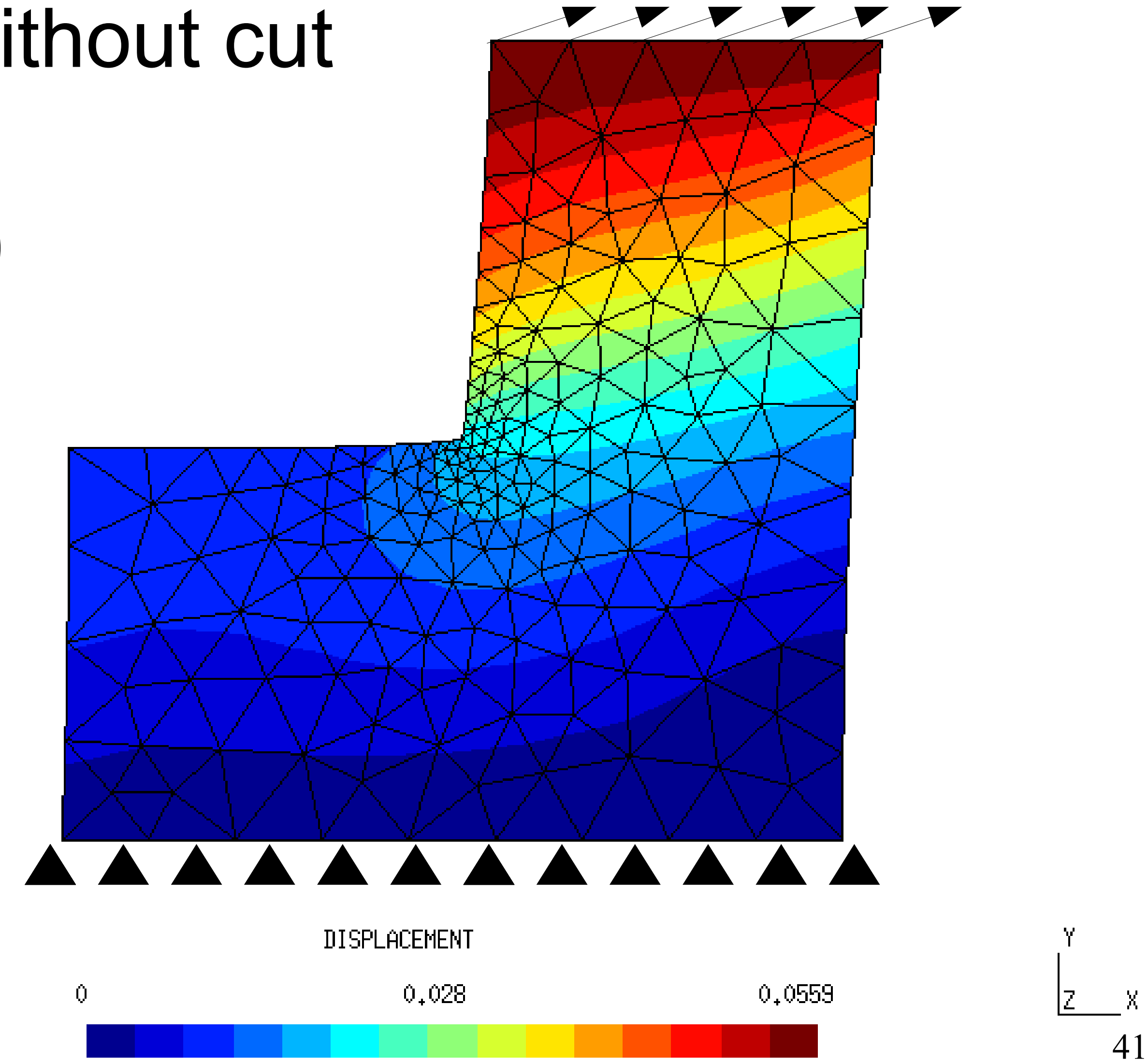


2D Example

- Displacements without cut (standard FEM)

  $\bar{u}(x) = \sum \lambda_i \cdot \bar{N}_i(x)$

The $\bar{N}_i(x)$ are the classical linear shape functions (order 1 Lagrange)



Extended Finite Elements

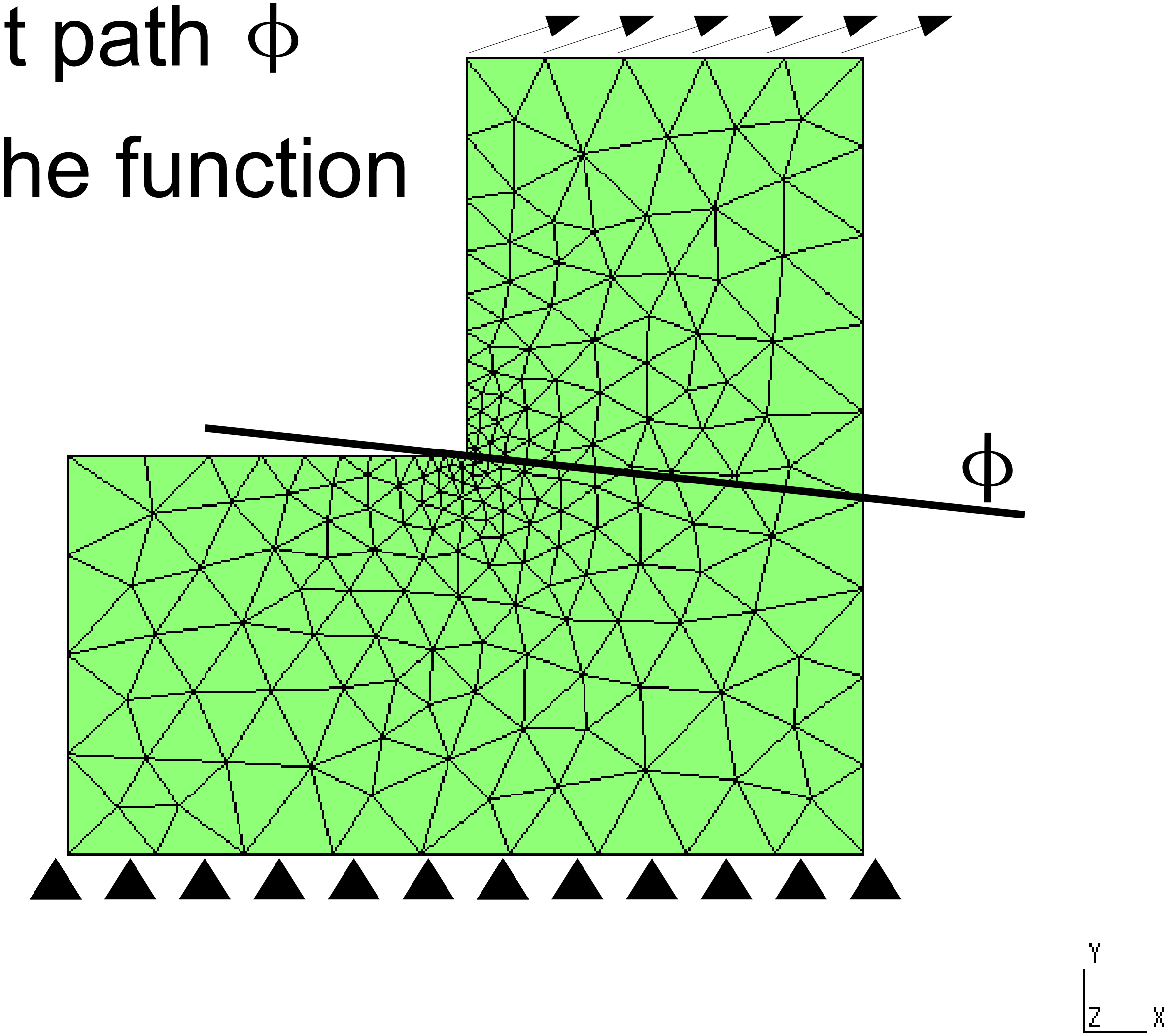
2D Example

- Lets impose a cut path ϕ
- Modifications of the function space :

$$\bar{u}(x) = \sum_{i \in \Omega} \lambda_i \cdot \bar{N}_i(x)$$

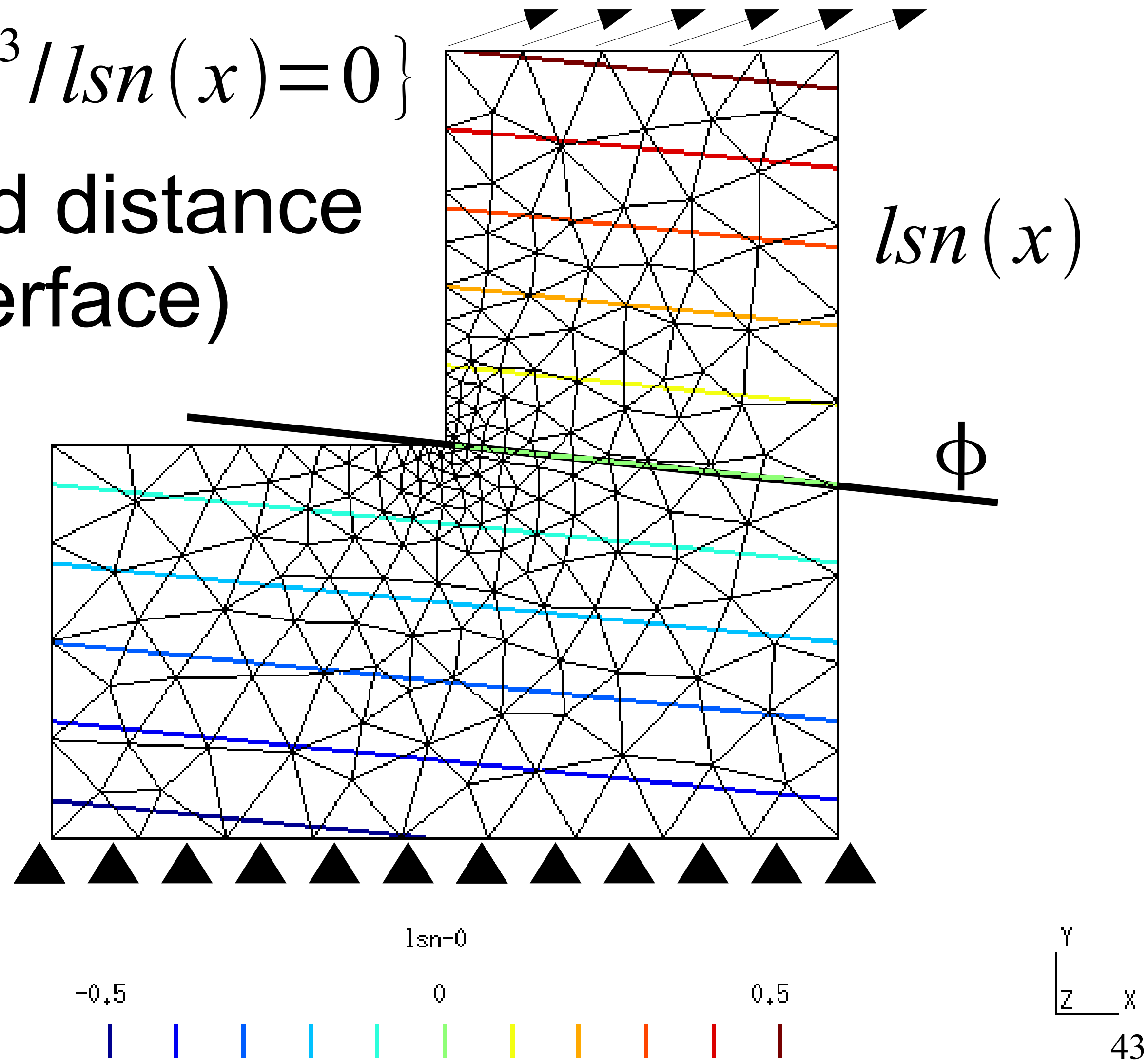
$$+ \sum_{i \in C} \lambda_i^* \cdot \bar{N}_i(x) \cdot H^*(s)$$

- How to define $H^*(s)$ and the set C ?



2D Example

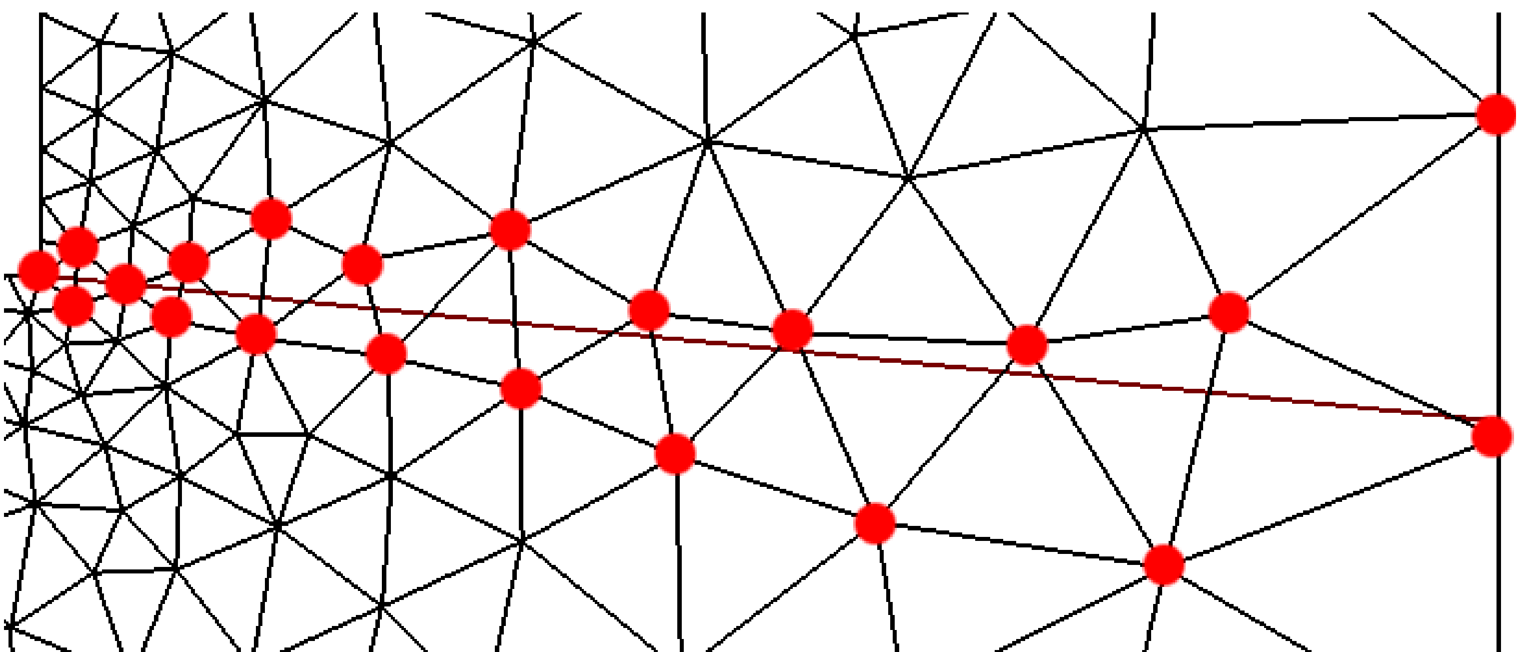
- The cutting path may be defined with a "level-set" ϕ
  We have $\phi = \{ x \in \mathbb{R}^3 / lsn(x) = 0 \}$
- $lsn(x)$ is the signed distance function (to the interface)
- One simply takes :
  $s = lsn(x)$
  $H^*(s) = H^*(lsn(x))$



Extended Finite Elements

2D Example

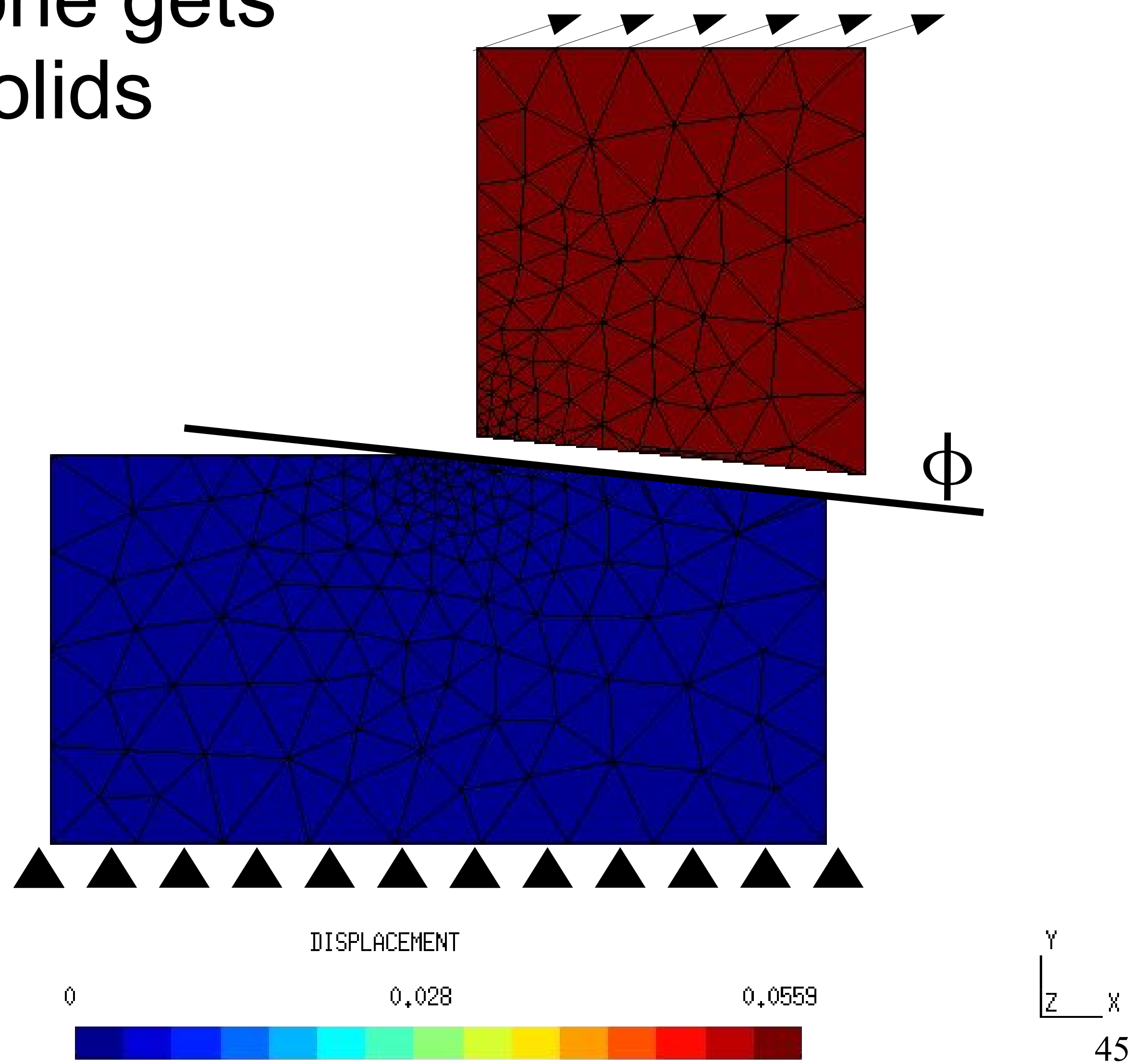
- Definition of the enriched degrees of freedom (the set $\mathcal{C}$)
  - Those are the nodes of the elements completely cut by ϕ (iso-0 of the level-set)



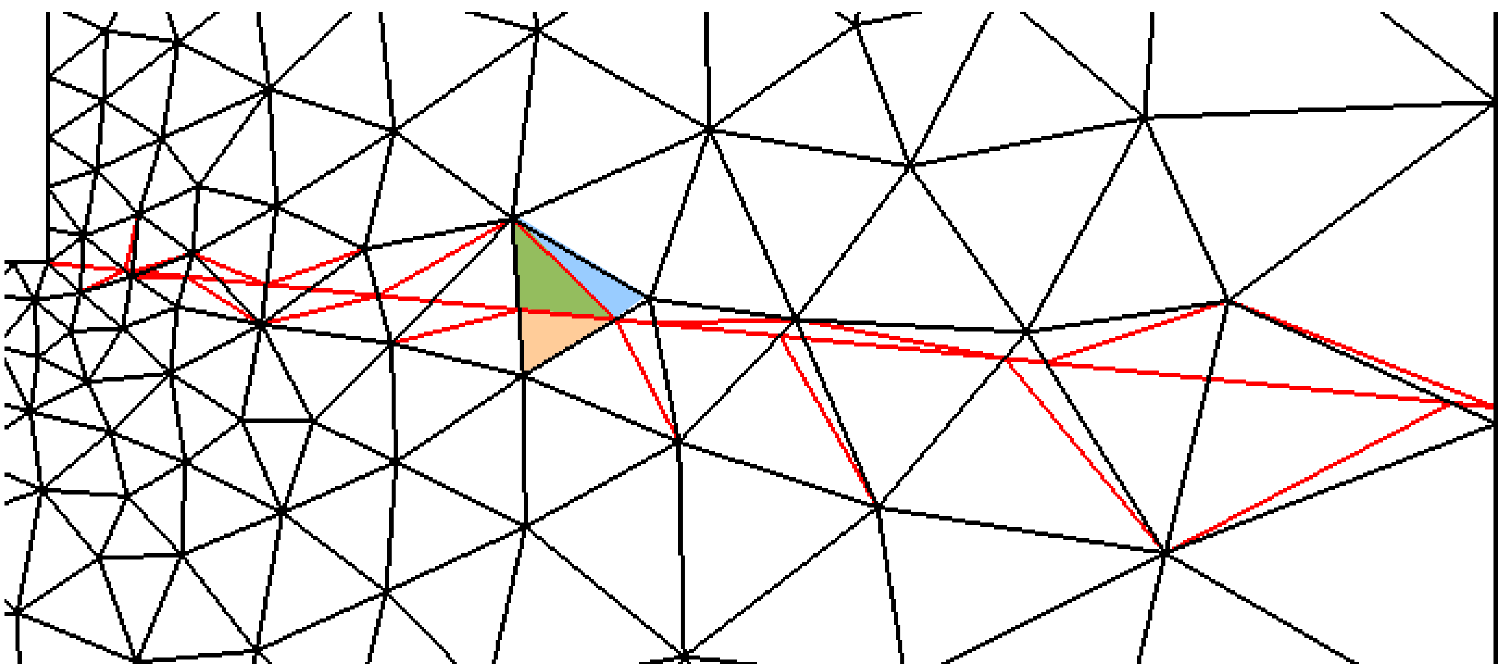
Extended Finite Elements

2D Example

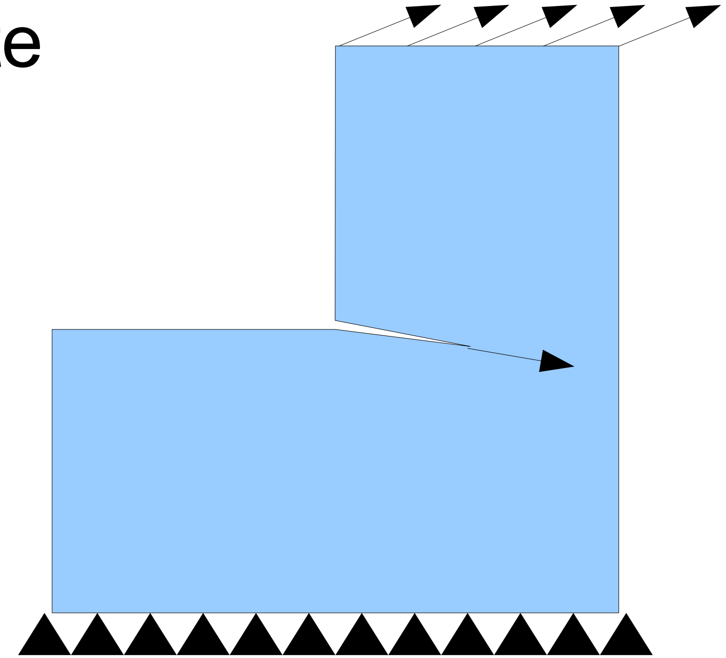
- After assembly and solving the linear system one gets two independent solids (as expected)
- The geometry of ϕ may be arbitrary.
- No need of any remeshing



- Integration
  - One need to subdivide elements that are cut by the interface (discontinuous functions to integrate)
    - On each sub triangle (in red here), a classical Gaussian quadrature is used because the integrand is a polynomial.



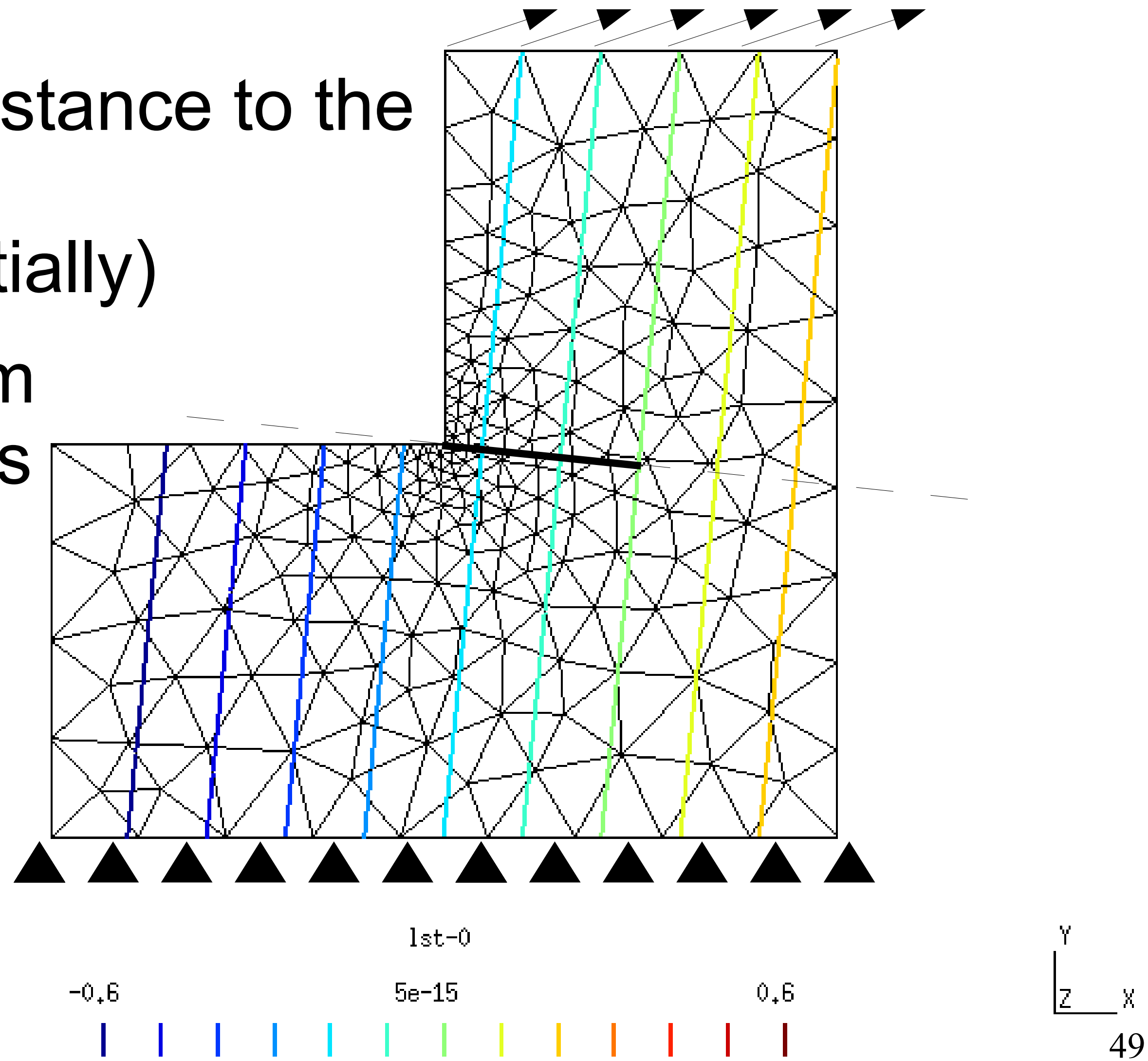
- Crack modelling
  - Historically, this is the first application of the extended finite element method
  - The crack propagates, and one does not want to generate a new mesh at each time step
  - A crack is in fact an incomplete cut made in the domain



Cracks

- Geometrical representation of the crack
  - It is not part of the mesh (by definition)
  - Its surface is therefore defined, as before, with a level set lsn that represents the normal distance to the surface.
  - One also needs the location where it stops (on its surface)
    - Crack tip (or front in 3D)

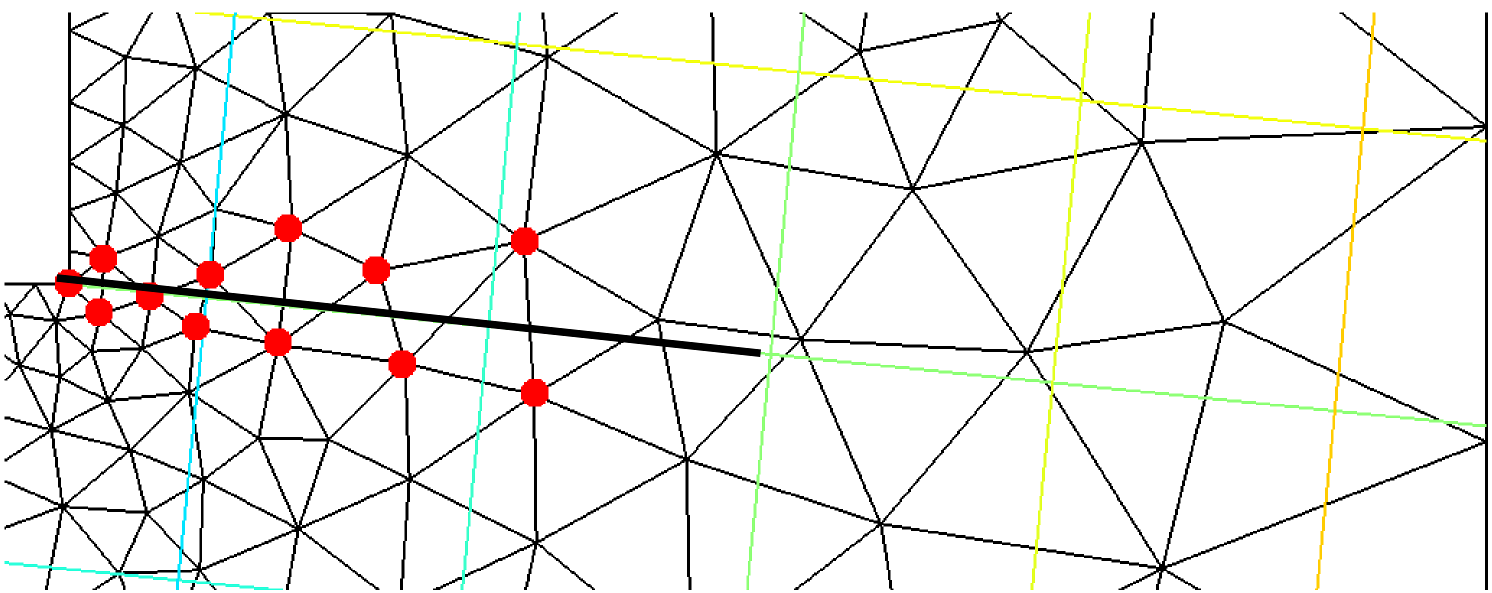
- We make use of another level set $lst(x)$
  - It represents the distance to the crack front (measured tangentially)
  - Both level sets form an orthogonal basis at the crack tip



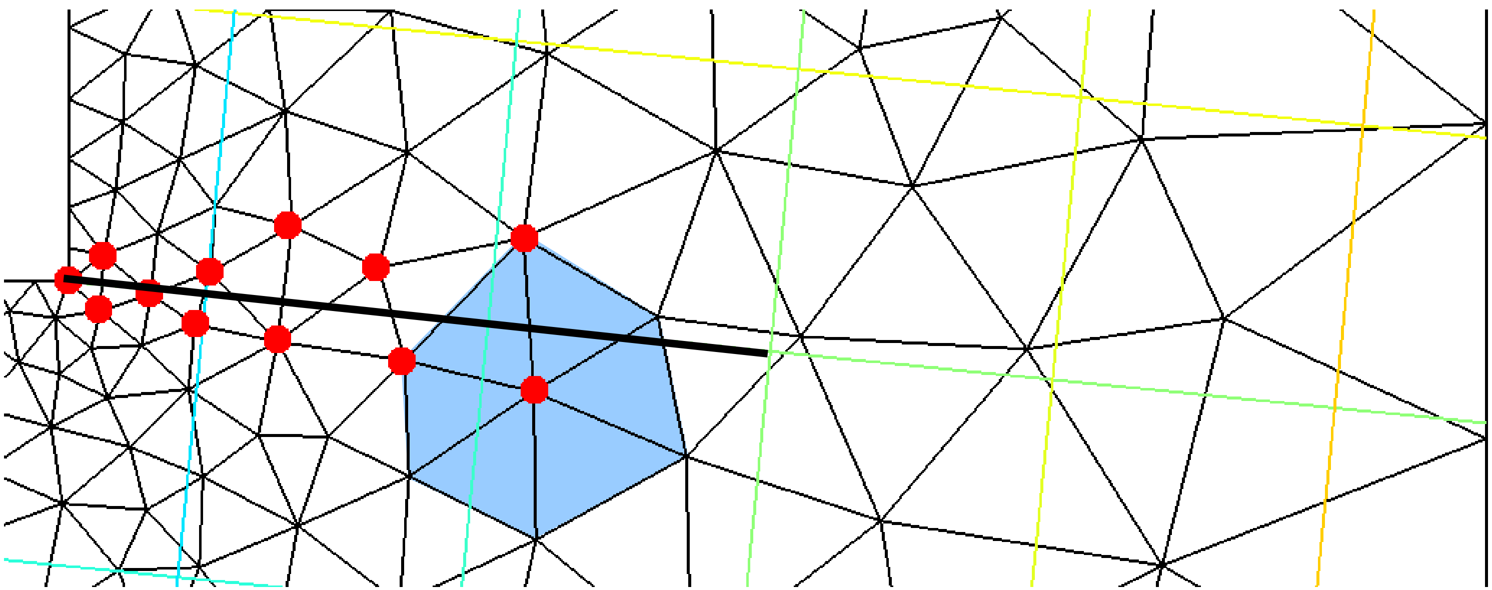
- The locus of the crack is therefore defined as :

$$\phi = \{ x \in \mathbb{R}^3 / lsn(x) = 0, lst(x) \leq 0 \}$$

- The enrichment set C is also modified :



- The enrichment set $\mathcal{C}$ is also modified :
  - Zone of influence of the new shape functions



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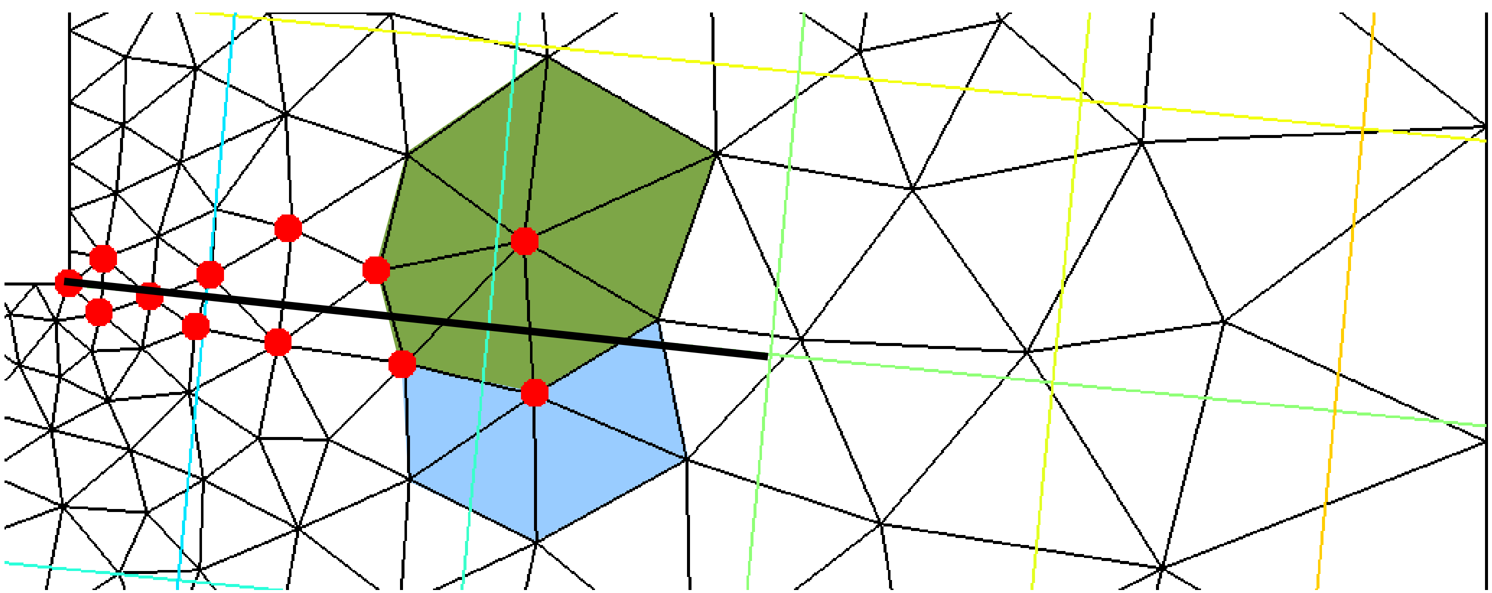
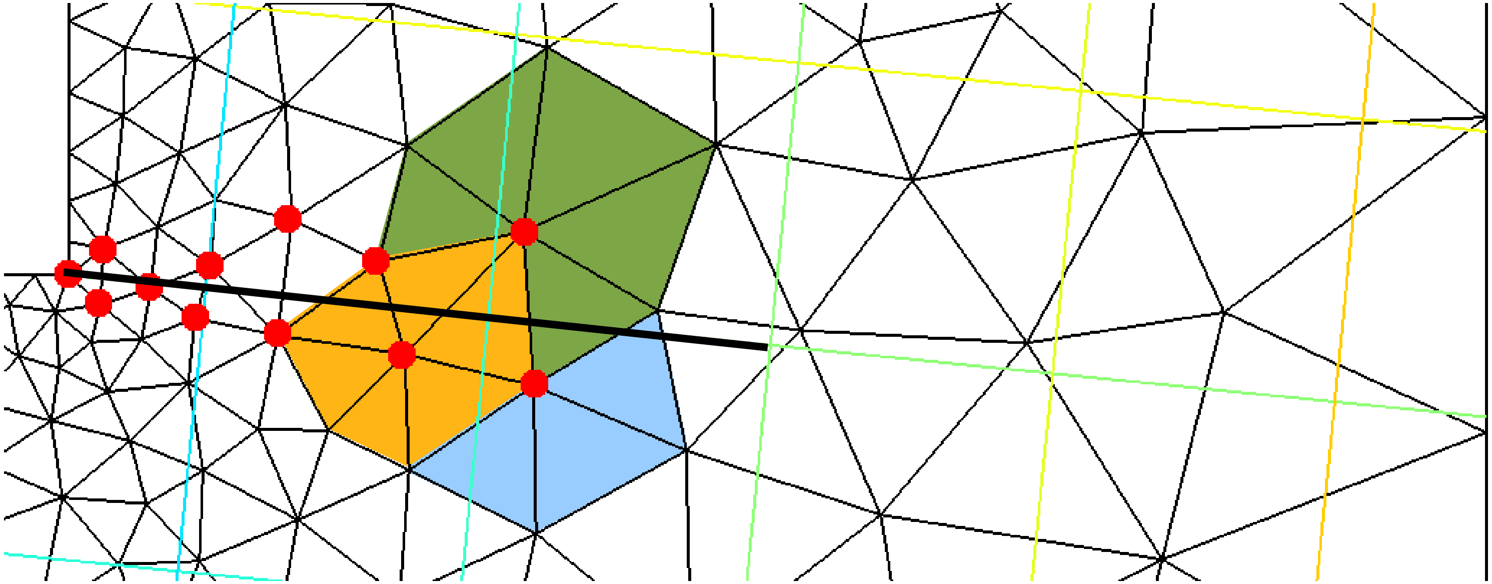
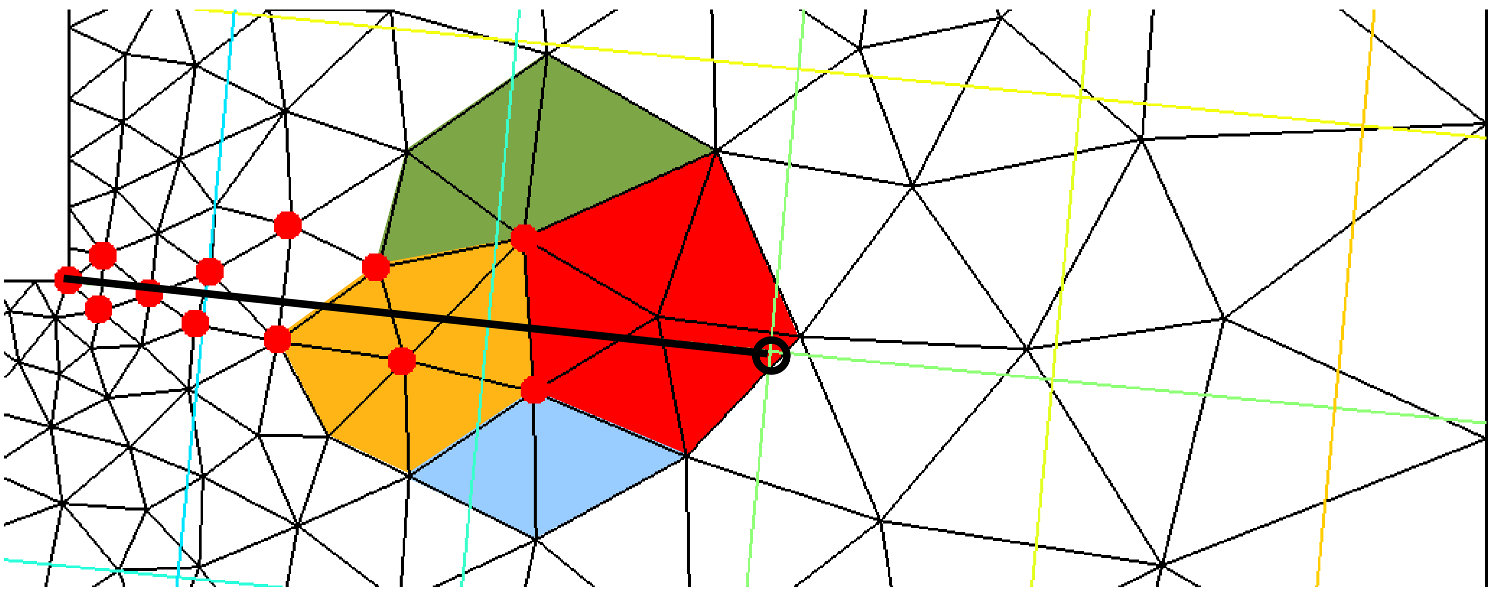


- The enrichment set $\mathcal{C}$ is also modified :
  - Zone of influence of the new shape functions
    - Either it cannot cover the crack until its tip or front...



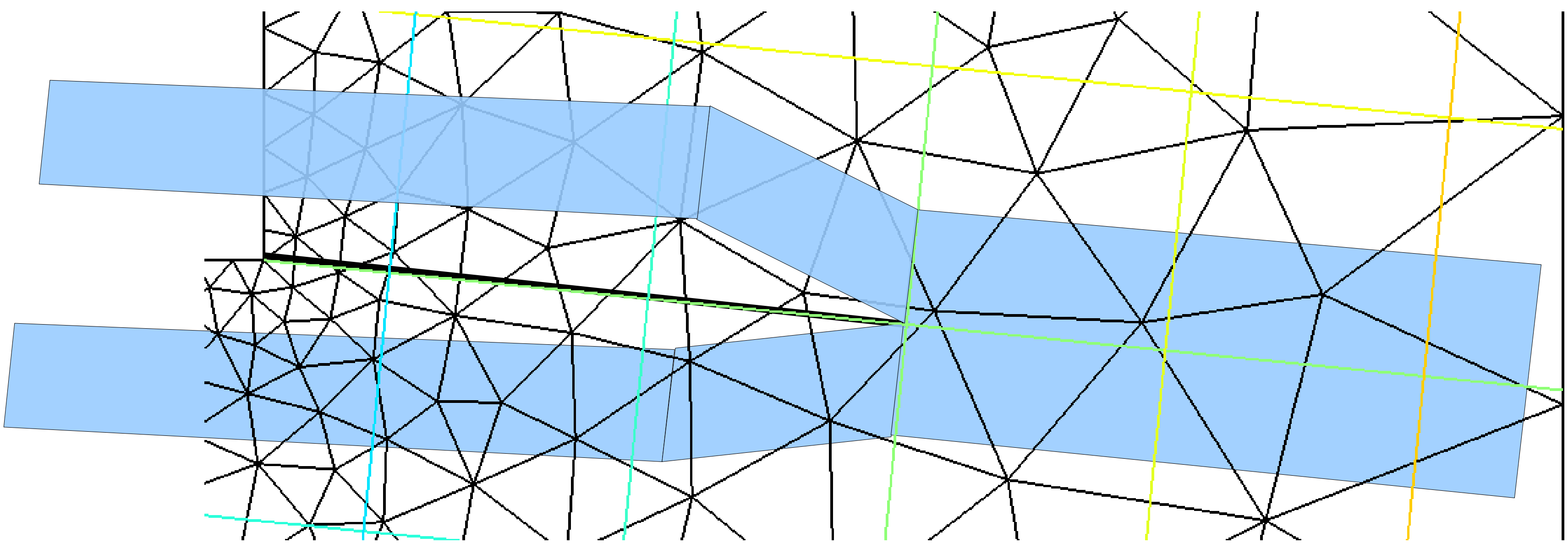
- The enrichment set $\mathcal{C}$ is also modified :
  - Zone of influence of the new shape functions
    - Either it cannot cover the crack until its tip or front... or it goes a bit too far



Extended Finite Elements

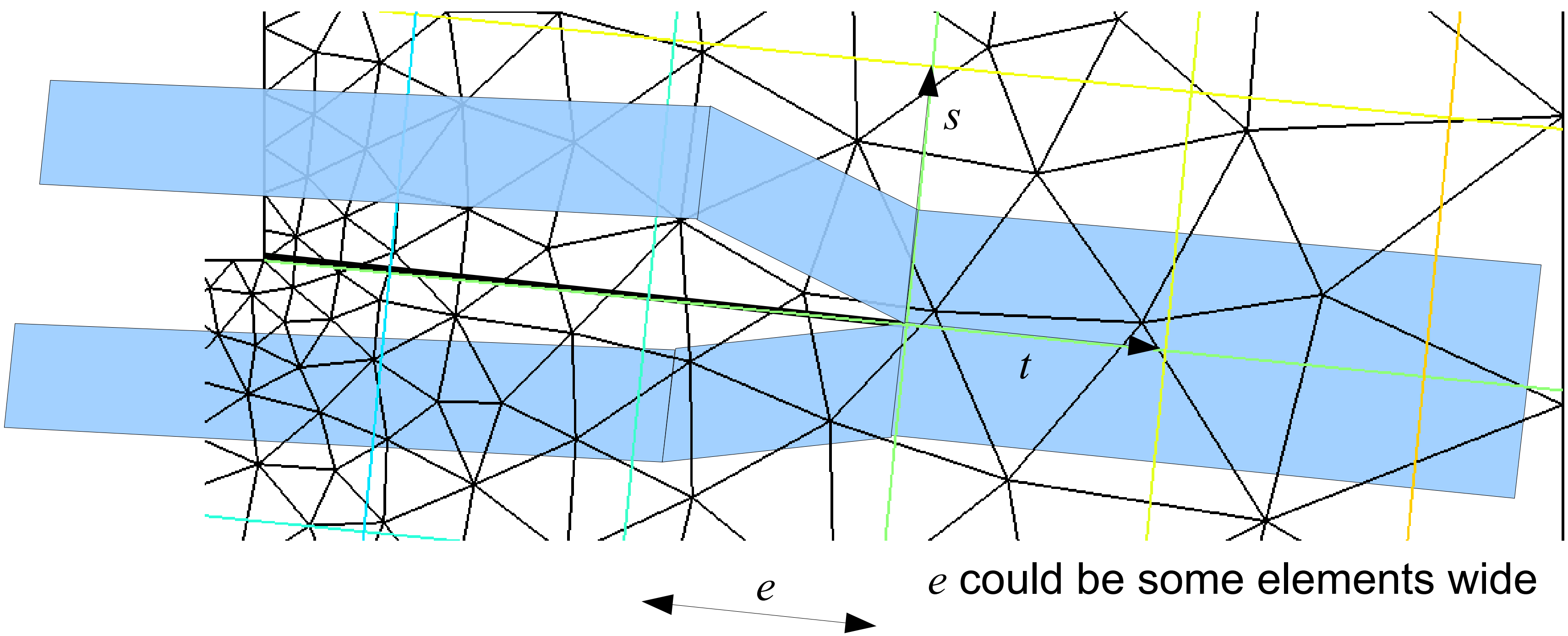
Cracks

- A special procedure is needed at the crack tip
  - The enrichment function should be discontinuous until the crack tip; continuous beyond.



Cracks

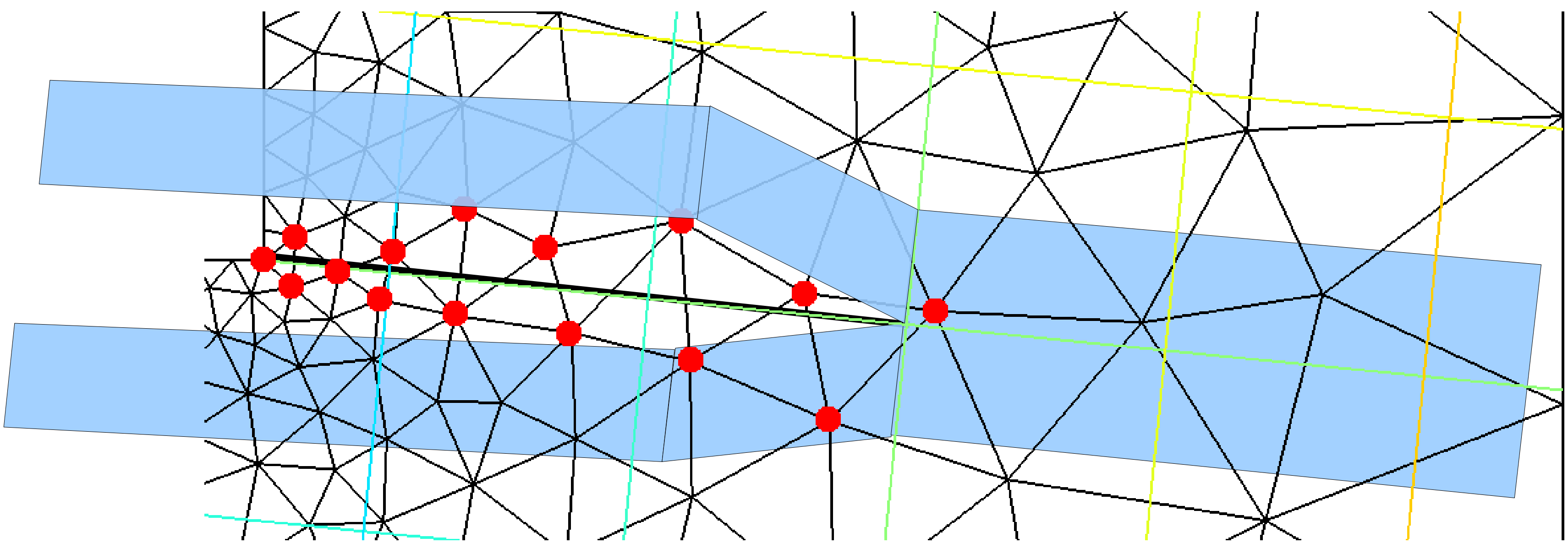
$$T(s,t)=\begin{cases}0 & \text{if } t\geq 0\\ H^{*}(s) & \text{if } t\leq -e\\ \dfrac{-t\,H^{*}(s)}{e} & \text{if } -e<t<0\end{cases} \quad \text{with} \quad \begin{cases} s=lsn(x)\\ t=lst(x)\end{cases}$$



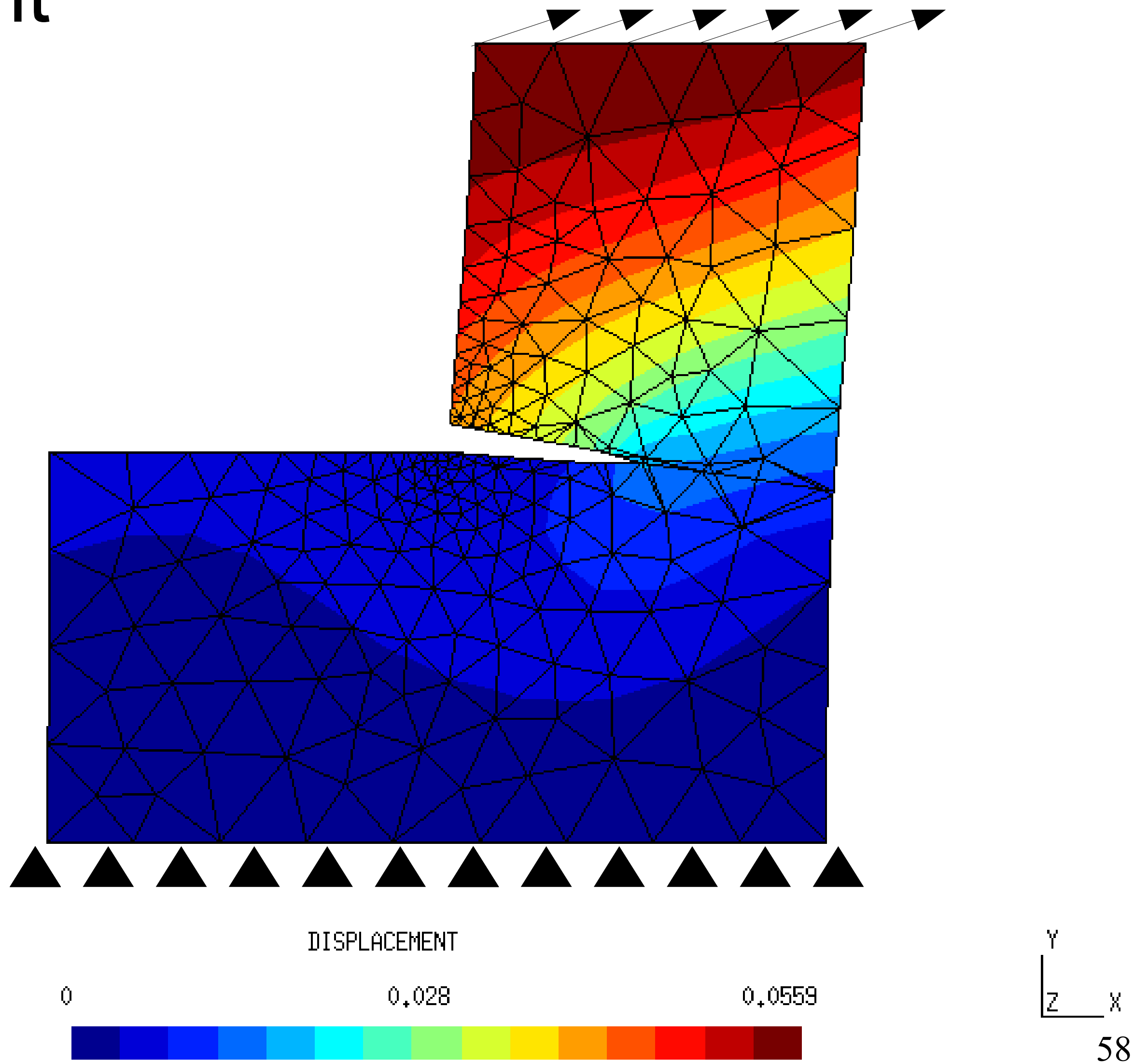
e could be some elements wide

- Alternate set of enriched elements C'
  - It includes every node for which the support is cut (at least partly) by the crack.

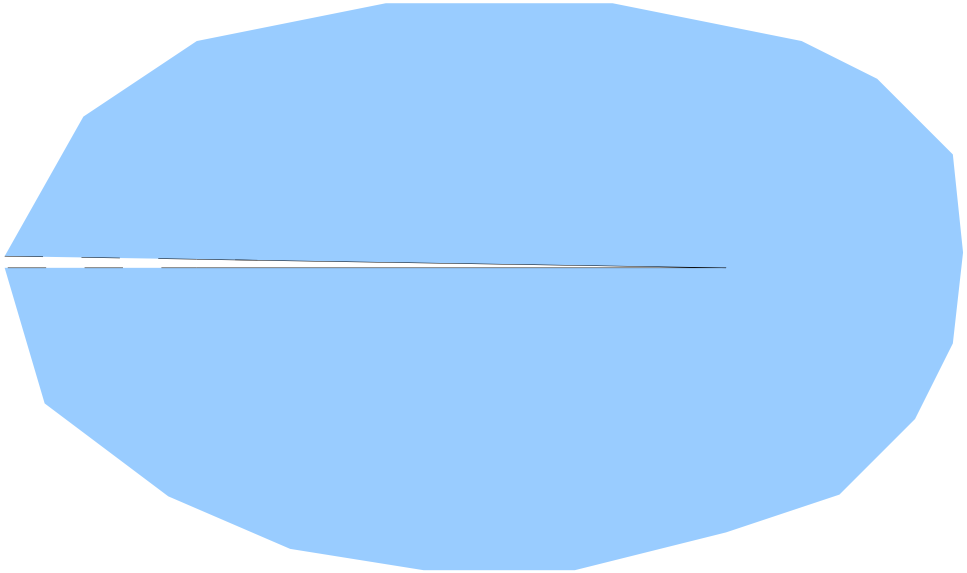
$$\bar{u}(x) = \sum_{i \in \Omega} \lambda_i \cdot \bar{N}_i(x) + \sum_{i \in \mathrm{C}'} \lambda_i^* \cdot \bar{N}_i(x) \cdot T(t, s)$$



- Displacements with a crack tip enrichment



- In fact, the form of the exact solution is known at the crack tip
  - Why not use this directly as a crack enrichment function ?
  - It is readily available for a crack in an infinite medium → see any good fracture mechanics course/book !



Extended Finite Elements

Cracks

- A polar basis is defined

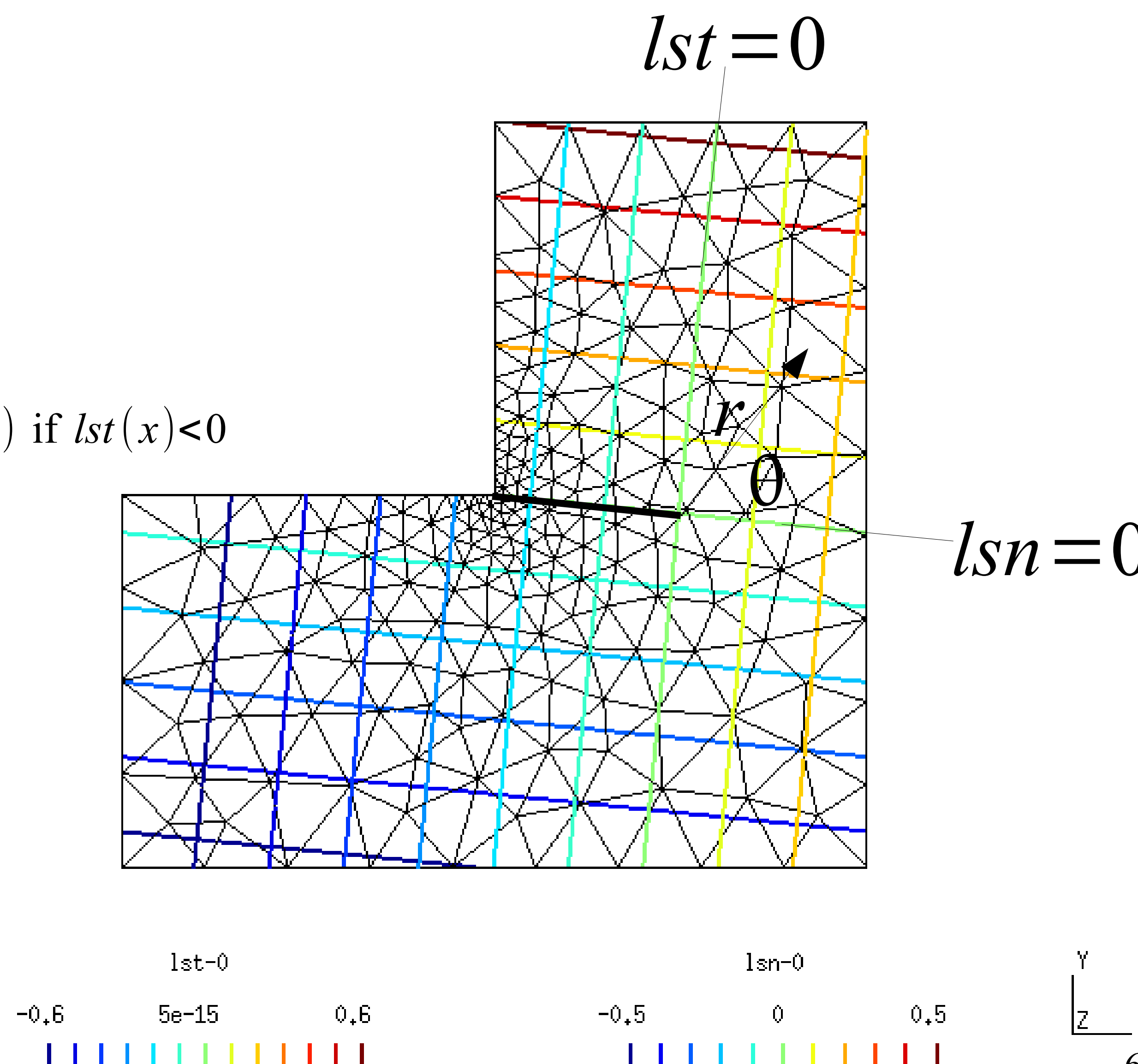
$$r = \sqrt{lsn(x)^2 + lst(x)^2}$$

$$\theta = \arg(\{lst(x), lsn(x)\})$$

$$\theta = \begin{cases} \arctan \dfrac{lsn(x)}{lst(x)} & \text{if } lst(x) > 0 \\ \left(\pi - \arctan \left| \dfrac{lsn(x)}{lst(x)} \right| \right) \operatorname{sgn} lsn(x) & \text{if } lst(x) < 0 \end{cases}$$

Or, better said :

$$\theta = \operatorname{atan2}(lsn(x), lst(x))$$



Extended Finite Elements

Cracks

- Exact asymptotic fields at the crack tip (crack in an infinite domain)

$$u_1 = \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_1 \cos\frac{\theta}{2}\left(\kappa - \cos\theta\right) + K_2 \sin\frac{\theta}{2}\left(\kappa + 2 + \cos\theta\right)\right\}$$

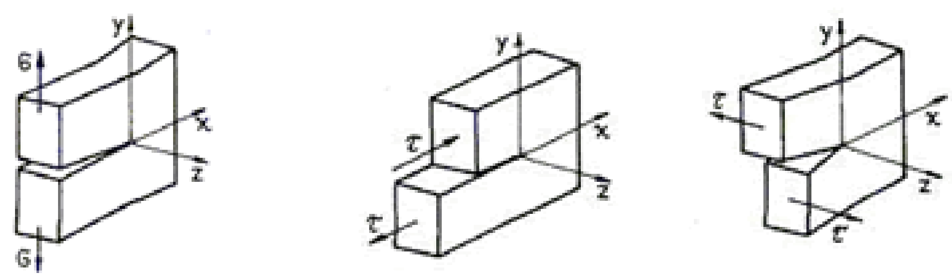
$$u_2 = \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_1 \sin\frac{\theta}{2}\left(\kappa - \sin\theta\right) + K_2 \cos\frac{\theta}{2}\left(\kappa - 2 + \cos\theta\right)\right\}$$

$$u_3 = \frac{2}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_3 \sin\frac{\theta}{2}\right\}$$

$$\mu = \frac{E}{2\left(1+\nu\right)}$$

$$\kappa = 3 - 4\nu$$

K_1 , K_2 and K_3 are constants which depend only on boundary conditions: they are called Stress Intensity Factors (SIFs)



Cracks

- Some analytical manipulations lead to :

$$u_1 = a_1\sqrt{r}\sin\frac{\theta}{2} + a_2\sqrt{r}\cos\frac{\theta}{2} + a_3\sqrt{r}\sin\frac{\theta}{2}\sin\theta + a_4\sqrt{r}\cos\frac{\theta}{2}\sin\theta + BC(x)$$

$$u_2 = b_1\sqrt{r}\sin\frac{\theta}{2} + b_2\sqrt{r}\cos\frac{\theta}{2} + b_3\sqrt{r}\sin\frac{\theta}{2}\sin\theta + b_4\sqrt{r}\cos\frac{\theta}{2}\sin\theta + BC(x)$$

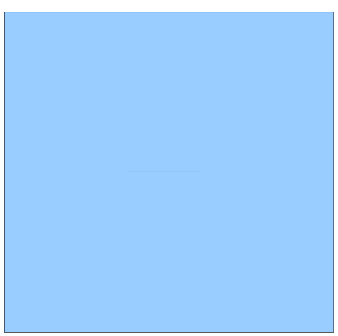
$$u_3 = c_1\sqrt{r}\sin\frac{\theta}{2} + c_2\sqrt{r}\cos\frac{\theta}{2} + c_3\sqrt{r}\sin\frac{\theta}{2}\sin\theta + c_4\sqrt{r}\cos\frac{\theta}{2}\sin\theta + BC(x)$$

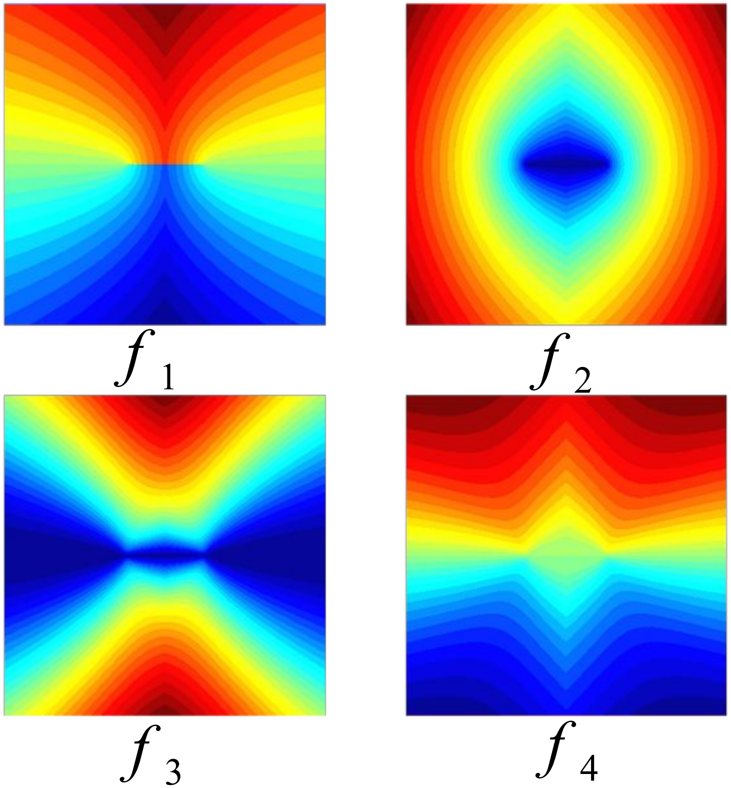
- One can therefore use only 4 enrichment functions (they span the whole function space)

$$\begin{cases} f_1 = \sqrt{r}\sin\frac{\theta}{2} & f_3 = \sqrt{r}\sin\frac{\theta}{2}\sin\theta \\ f_2 = \sqrt{r}\cos\frac{\theta}{2} & f_4 = \sqrt{r}\cos\frac{\theta}{2}\sin\theta \end{cases}$$

- One may notice that only f_1 is discontinuous; the others are simply C^0.

- Shape of the enrichment functions in the case of an Irwin crack



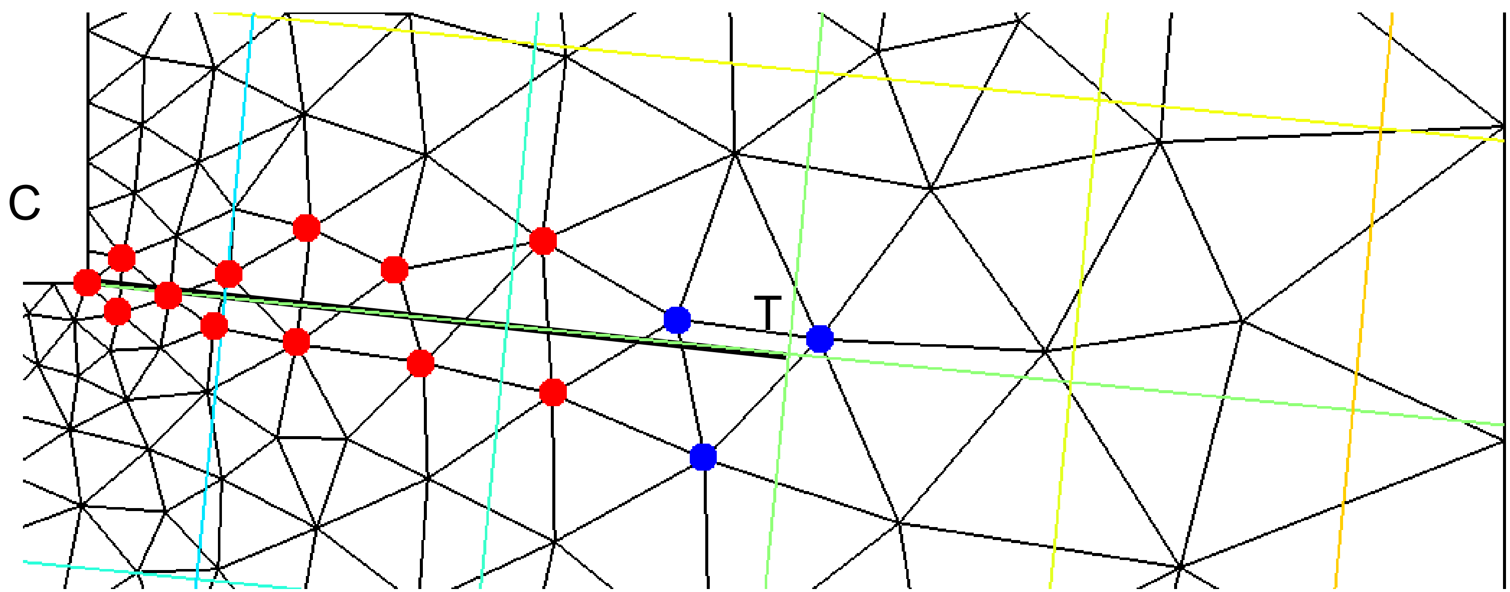


- A new function space

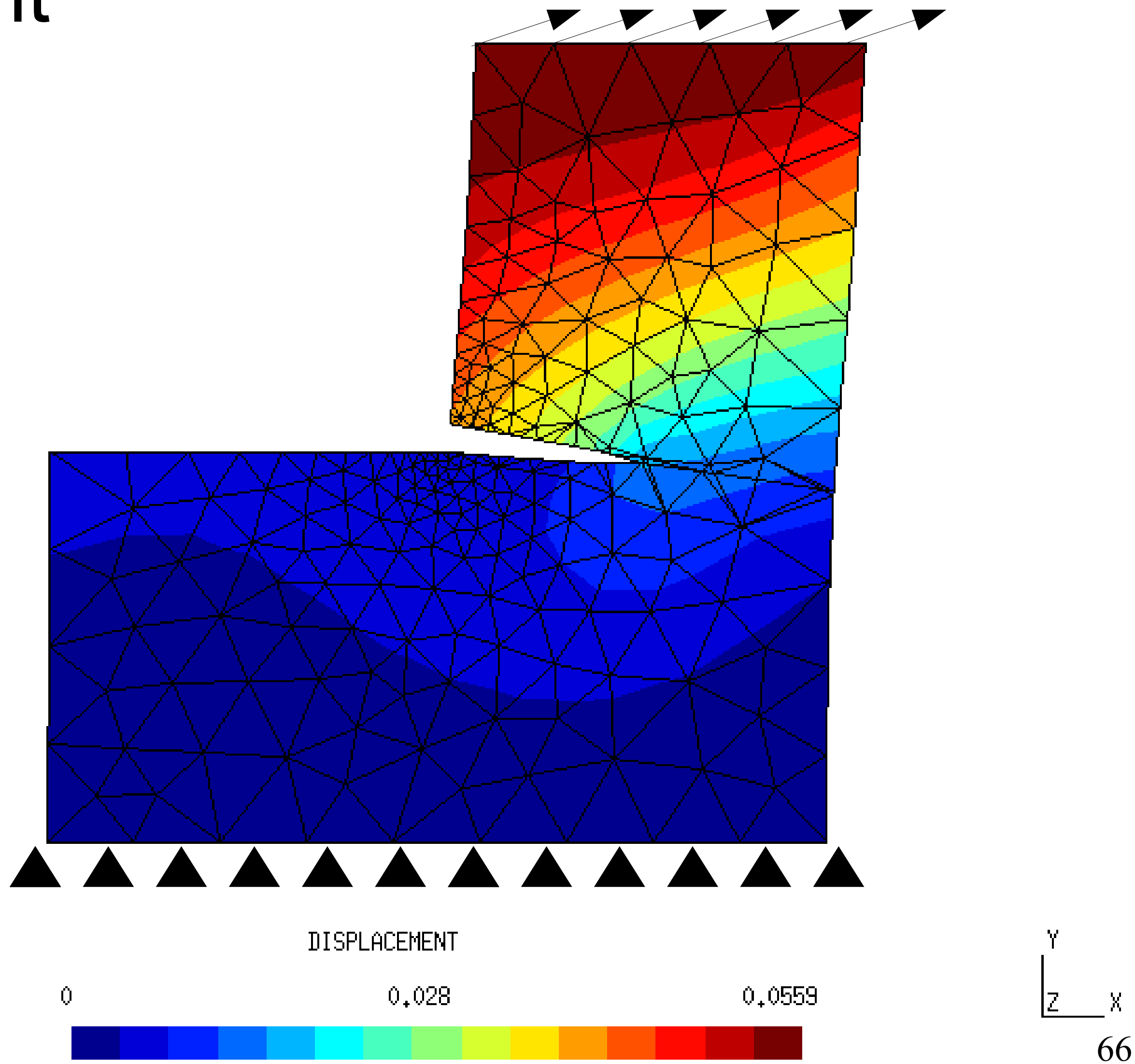
$$\bar{u}(x) = \sum_{i\in\Omega} \lambda_i \cdot \bar{N}_i(x) + \sum_{i\in C} \lambda_i^* \cdot \bar{N}_i(x) \cdot H^*(s) + \sum_{i\in T} \sum_{j\in 1..4} \lambda_i^j \cdot \bar{N}_i(x) \cdot f_j(r,\theta)$$

- Where to enrich ?
  - At the crack tip (T), because the rest of the cracked domain is already covered by the Heaviside enrichment
  - The analytical solution used to build the $f_j(r,\theta)$ is only valid around the crack tip.

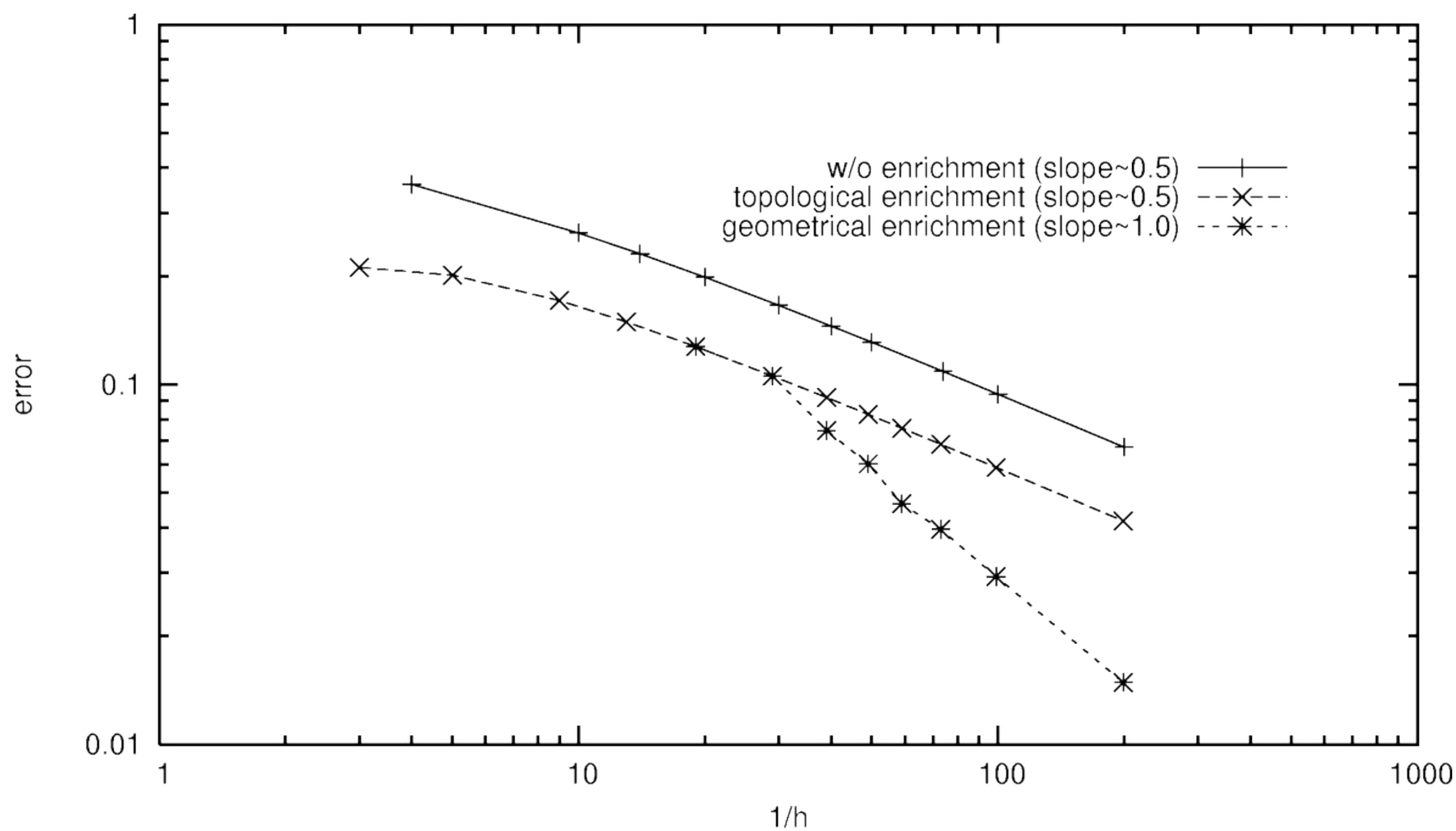
- Choice of the nodes to enrich
  - The set C contains nodes for which the support is completely cut by the crack
  - The set T contains the nodes for which the support contains or touches the crack tip



- Displacements with the new crack tip enrichment



- If one chooses a good enrichment procedure, one may get a better convergence rate than observed with regular finite elements.



Cracks

- To be able to propagate a crack, it is needed to :

  Perform the assembly of the linear system

  Solve the linear system

  Compute adequate propagation parameters

  Update level-sets lsn and lst

- Crack propagation obeys to well defined physical laws
  - Fatigue
  - Fragile fracture etc...

- What are the adequate parameters of crack propagation
  - Charge coefficients (stress intensity factors) that are linked to the geometry of the problem and the boundary conditions.
    - Intrisic parameters having effects on the material just in front of the crack path.
  - Material behaviour with respect to these SIFs : ductile propagation (mild steel) or fragile (glass, cast iron)
  - For ductile fracture, one often uses the ratio (number of loading cycle) w.r. to (crack advance)

- Computation of the stress intensity factors
  - Depend only on stress field around the crack
  - J integrals and interaction integrals

    (do not recall these, see further)

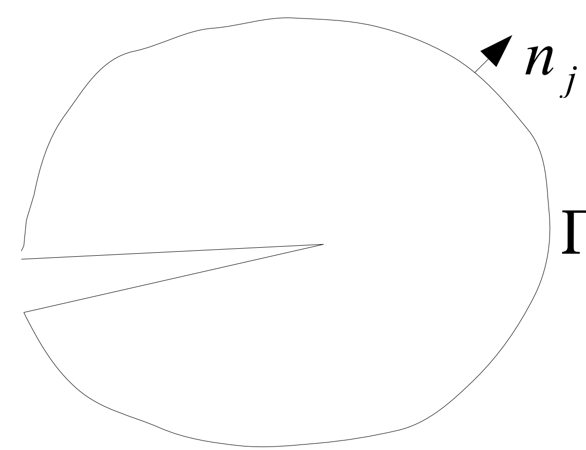
$$J=\int_\Gamma \left[\frac{1}{2}\sigma_{ij}\epsilon_{ij}\delta_{1j}-\sigma_{ij}\frac{\partial u_i}{\partial x_1}\right]n_j\,d\Gamma$$

$$J^{(1+2)}=\int_\Gamma \left[\frac{1}{2}(\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)})(\epsilon_{ij}^{(1)}+\epsilon_{ij}^{(2)})\delta_{1j}-(\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)})\frac{\partial(u_i^{(1)}+u_i^{(2)})}{\partial x_1}\right]n_j\,d\Gamma$$

$$=J^{(1)}+J^{(2)}+I^{(1+2)}$$

$$I^{(1+2)}=\int_\Gamma \left[\sigma_{ij}^{(1)}\epsilon_{ij}^{(2)}\delta_{1j}-\sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x_1}-\sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x_1}\right]n_j\,d\Gamma$$

$$I^{(1+2)}=2\frac{(1-\nu^2)}{E}(K_1^{(1)}K_1^{(2)}+K_2^{(1)}K_2^{(2)})+\frac{1}{\mu}K_3^{(1)}K_3^{(2)}$$

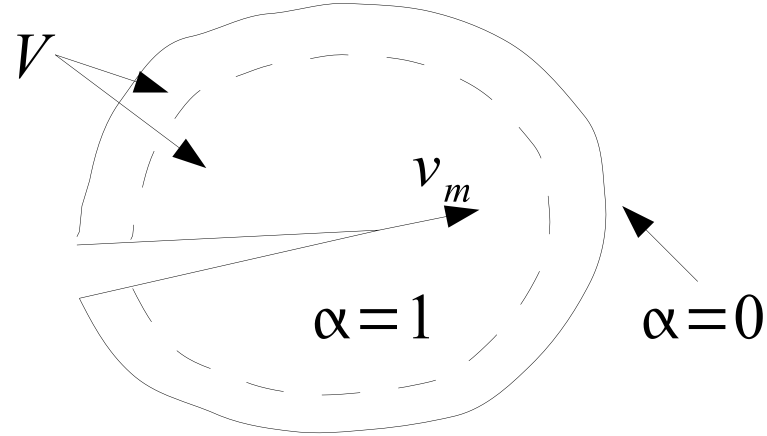


Cracks

- Going from a contour integral to a volume integral (unloaded crack)

$$I^{(1+2)}=\int_V \frac{\partial q_m}{\partial x_j}\left(\sigma_{kl}^{(1)}\epsilon_{kl}^{(2)}\delta_{mj}-\sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x_m}-\sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x_m}\right)dV$$

One have $q_m=\alpha\cdot v_m$ and α is equal to 1 inside the domain and vaniqhes on the boundary Γ.
v_m is the virtual crack propagation speed (norm=1)
One interpolates α on the mesh.



Cracks

- The interaction integrals allows to compute the stress intensity factors
  - Robust
  - Same good properties as the J- integral
  - See fracture mechanics course(s) for more info.

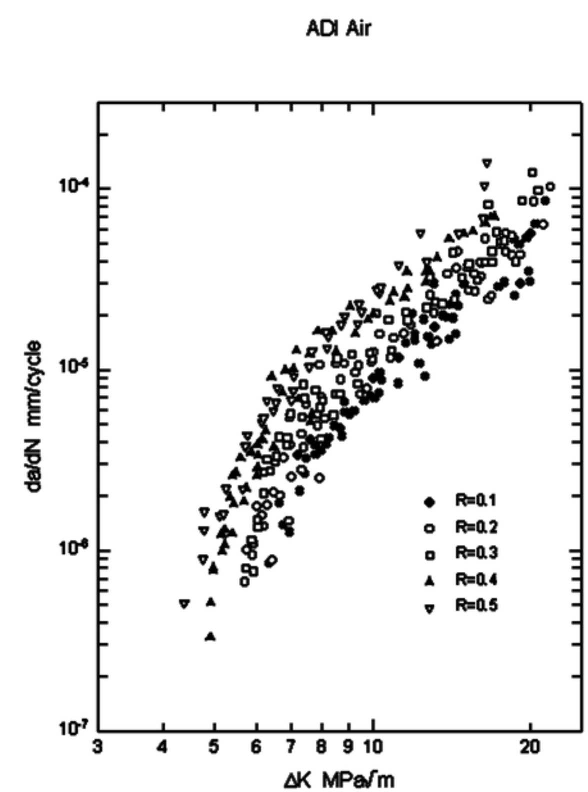
- Propagation speed

  Example : Alloys under cyclic loadings

  Paris law for the speed of propagation :

$$\frac{da}{dN} = C \cdot \Delta K^m$$

Alloy	m	C (m/cycle)
Steel	3	10^{-11}
Aluminium	3	10^{-12}
Nickel	3.3	$4 \cdot 10^{-12}$
Titanium	5	10^{-11}



- Direction is along the maximal tangent stress $\sigma_{\theta\theta}$

$$\begin{Bmatrix} \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \frac{K_1}{4\sqrt{2\pi r}} \begin{Bmatrix} 3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \\ \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \end{Bmatrix} + \frac{K_2}{4\sqrt{2\pi r}} \begin{Bmatrix} -3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2} \\ \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} \end{Bmatrix}$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad \rightarrow \quad \cos\frac{\theta_c}{2}\left[\frac{1}{2}K_1 \sin\theta_c + \frac{1}{2}K_2\left(3\cos\theta_c - 1\right)\right] = 0$$

$$\theta_c = 2\arctan\frac{1}{4}\left(\frac{K_1}{K_2} \pm \sqrt{\left(\frac{K_1}{K_2}\right)^2 + 8}\right)$$

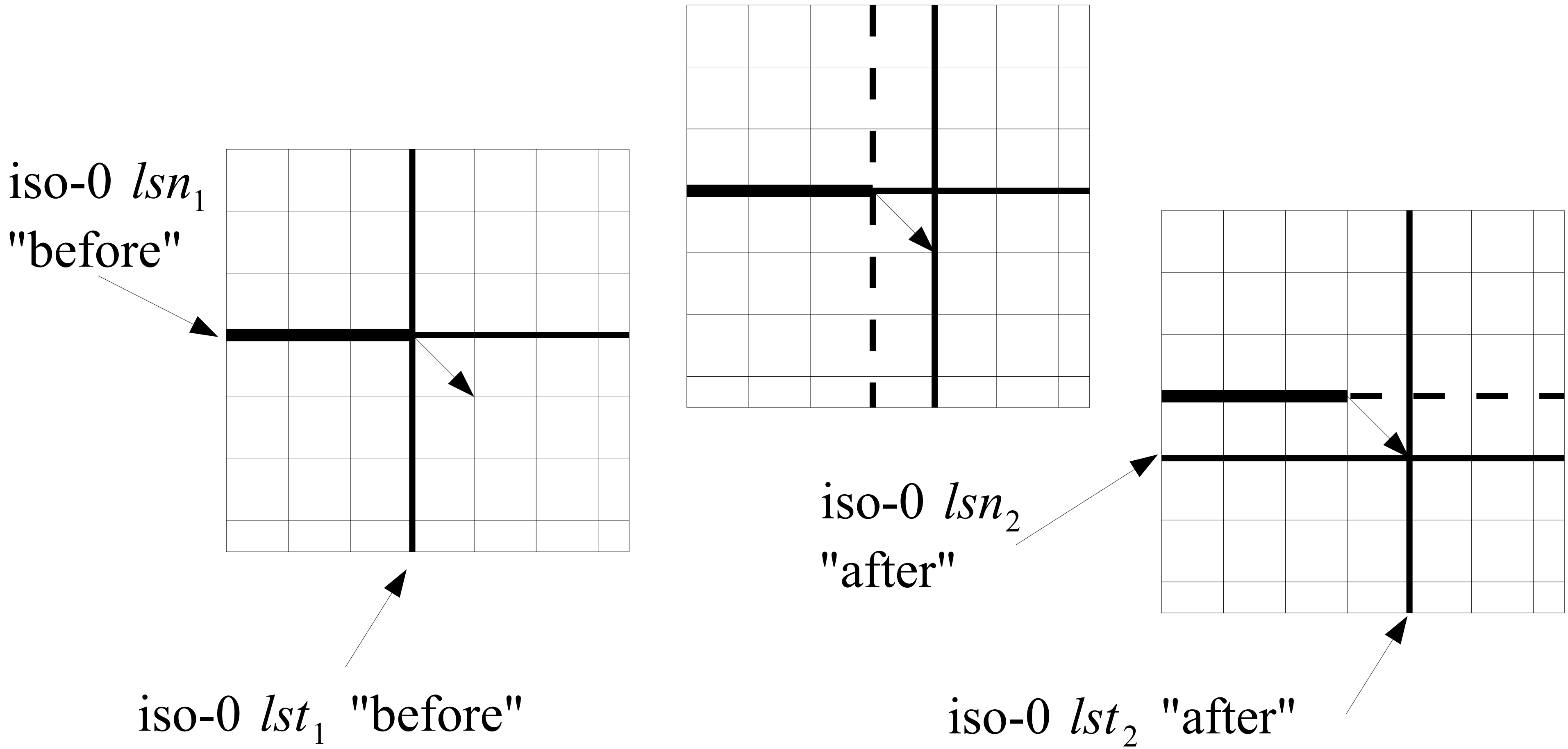
- One chooses θ_c that correspond to a maximal value of $\sigma_{\theta\theta}$ (in traction)

Level set update

- There exists many algorithms but the essential part is to :
  - Conserve the notion of signed distance function at the interface for lsn
  - Have an orthonormed frame in the vincinity of the crack tip (lst,lsn)

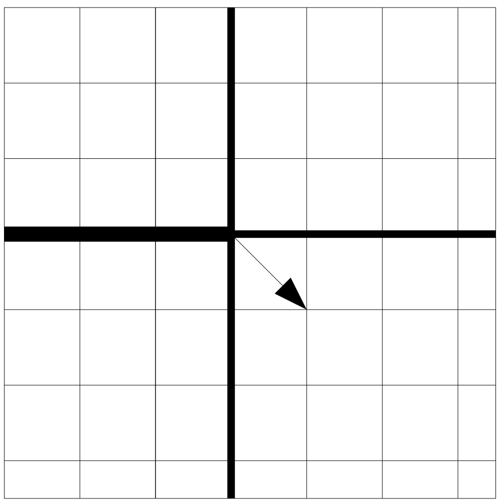
Level set update

- Transport of *lsn* and *lst*



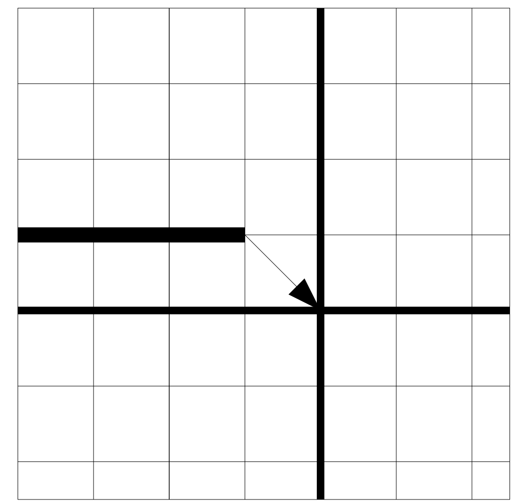
Level set update

- Rebuilding of lsn and lst



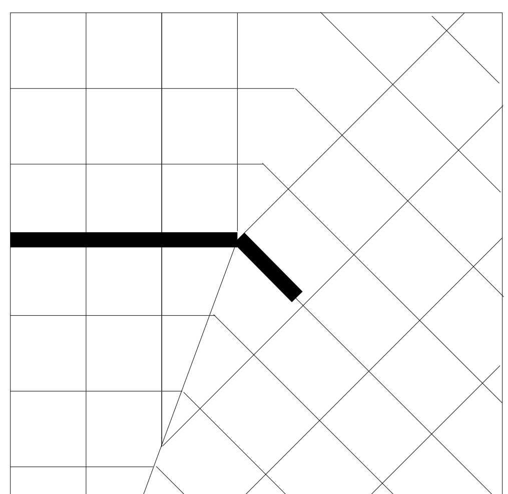
lsn_1 & lst_1 "before"

$$dx = lst_1 - lst_2$$
$$dy = lsn_1 - lsn_2$$
$$\alpha = \mathrm{atan2}(dy, dx)$$



lsn_2 & lst_2 "after"

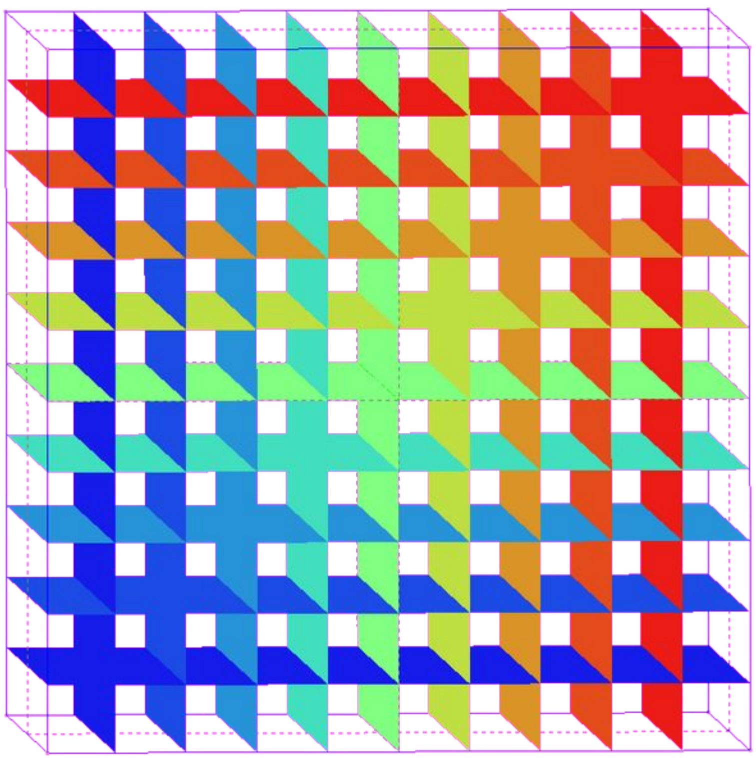
$$lst = lst_2$$
$$lsn = lsn_1$$

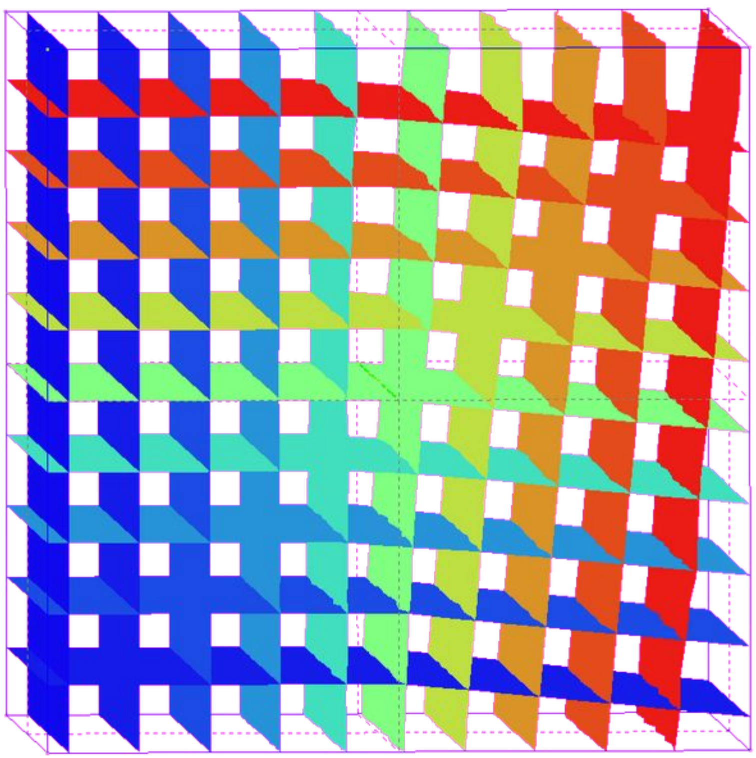


$$lst = \cos(\alpha) \cdot lst_2 + \sin(\alpha) \cdot lsn_2$$
$$lsn = -\sin(\alpha) \cdot lst_2 + \cos(\alpha) \cdot lsn_2$$

Extended Finite Elements

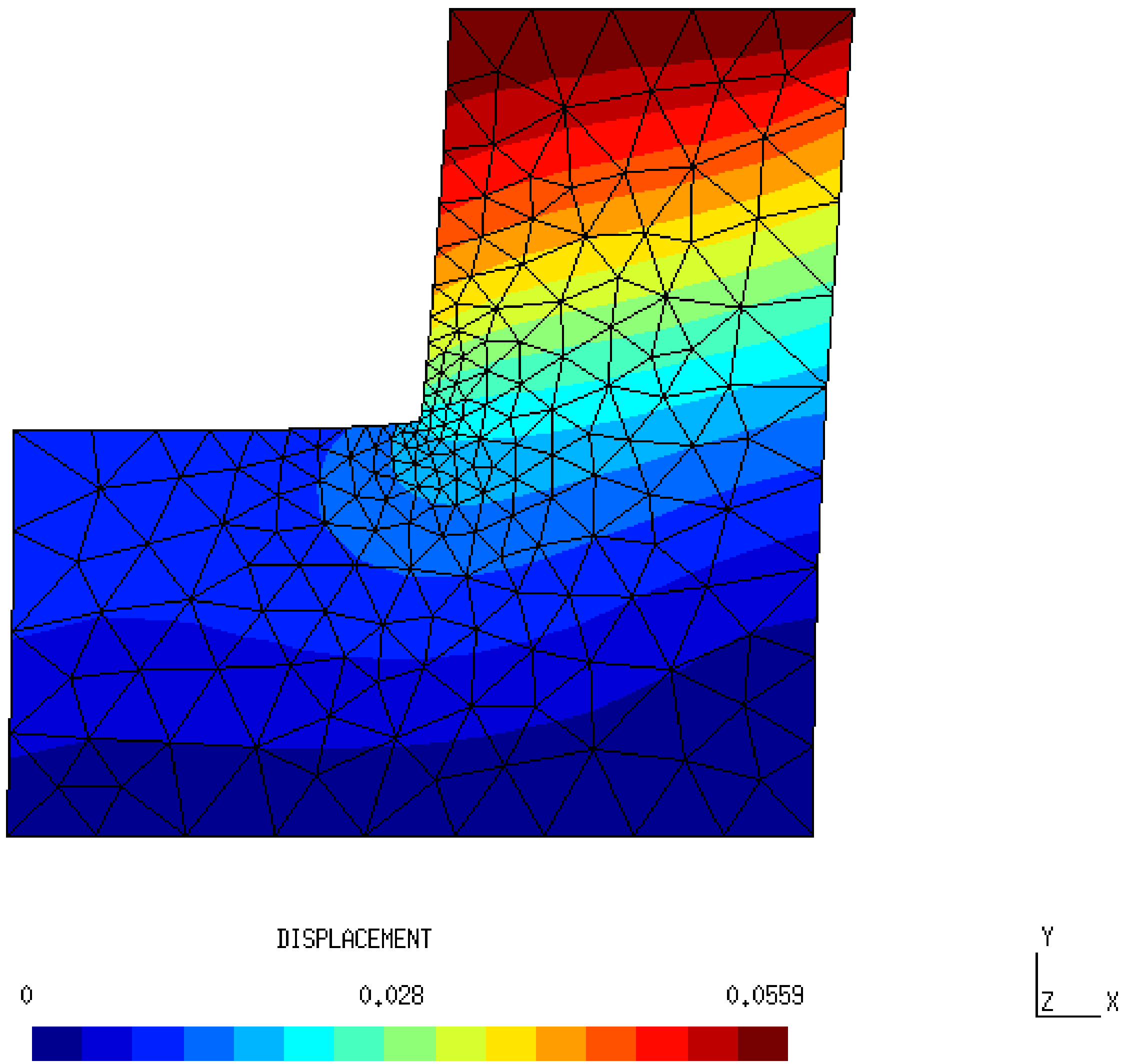
Level set update



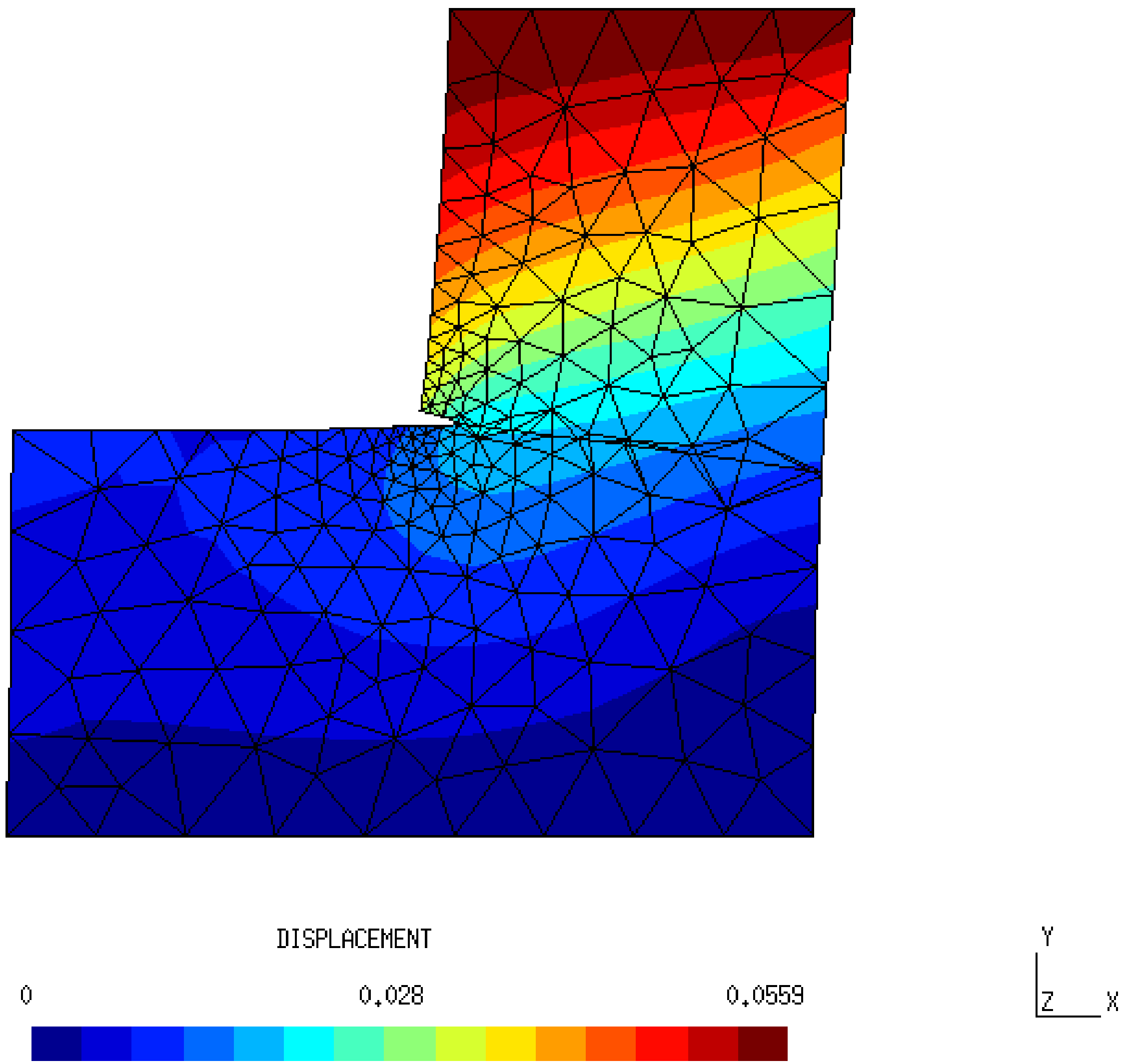


Extended Finite Elements

Propagation

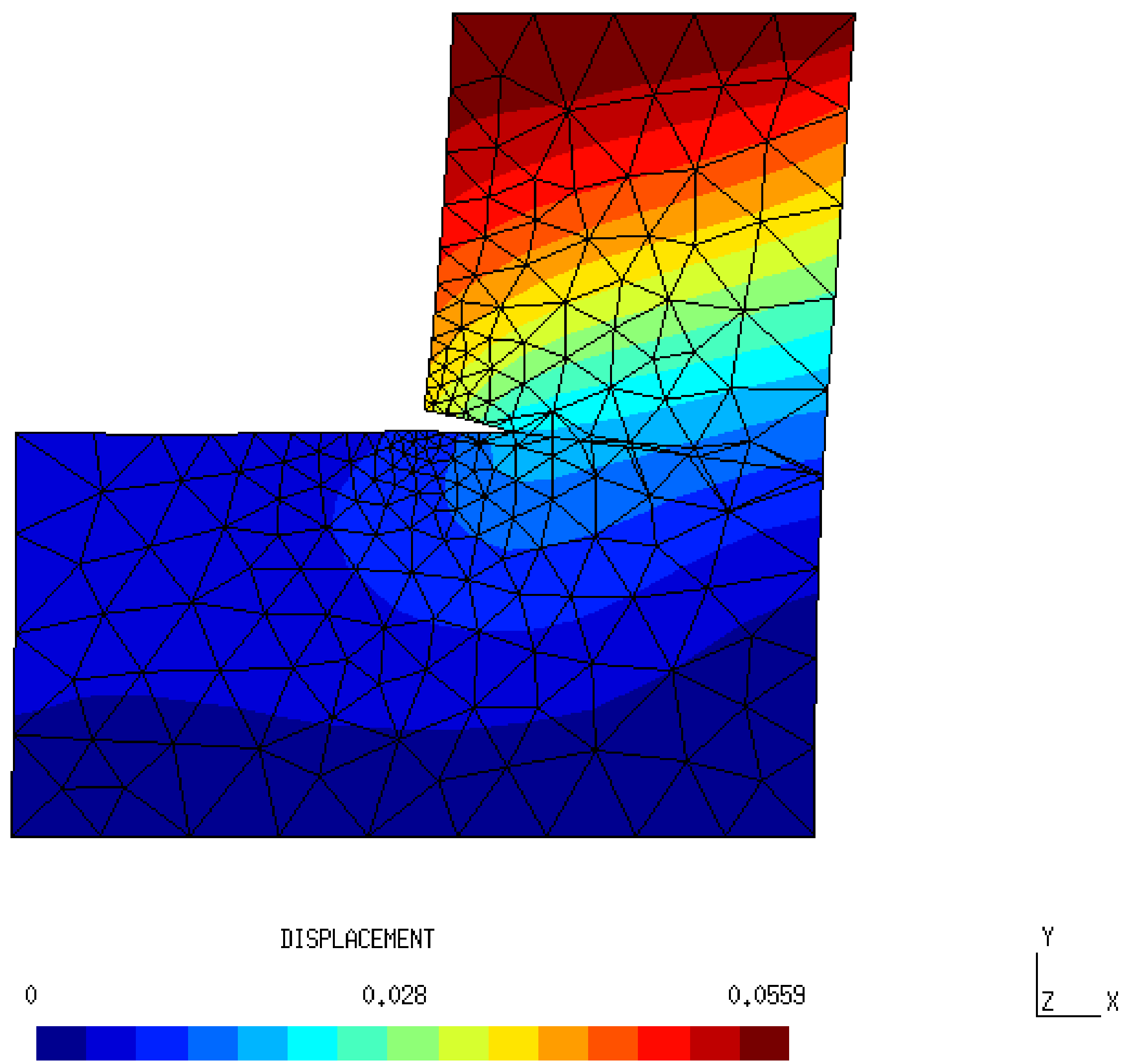


Propagation



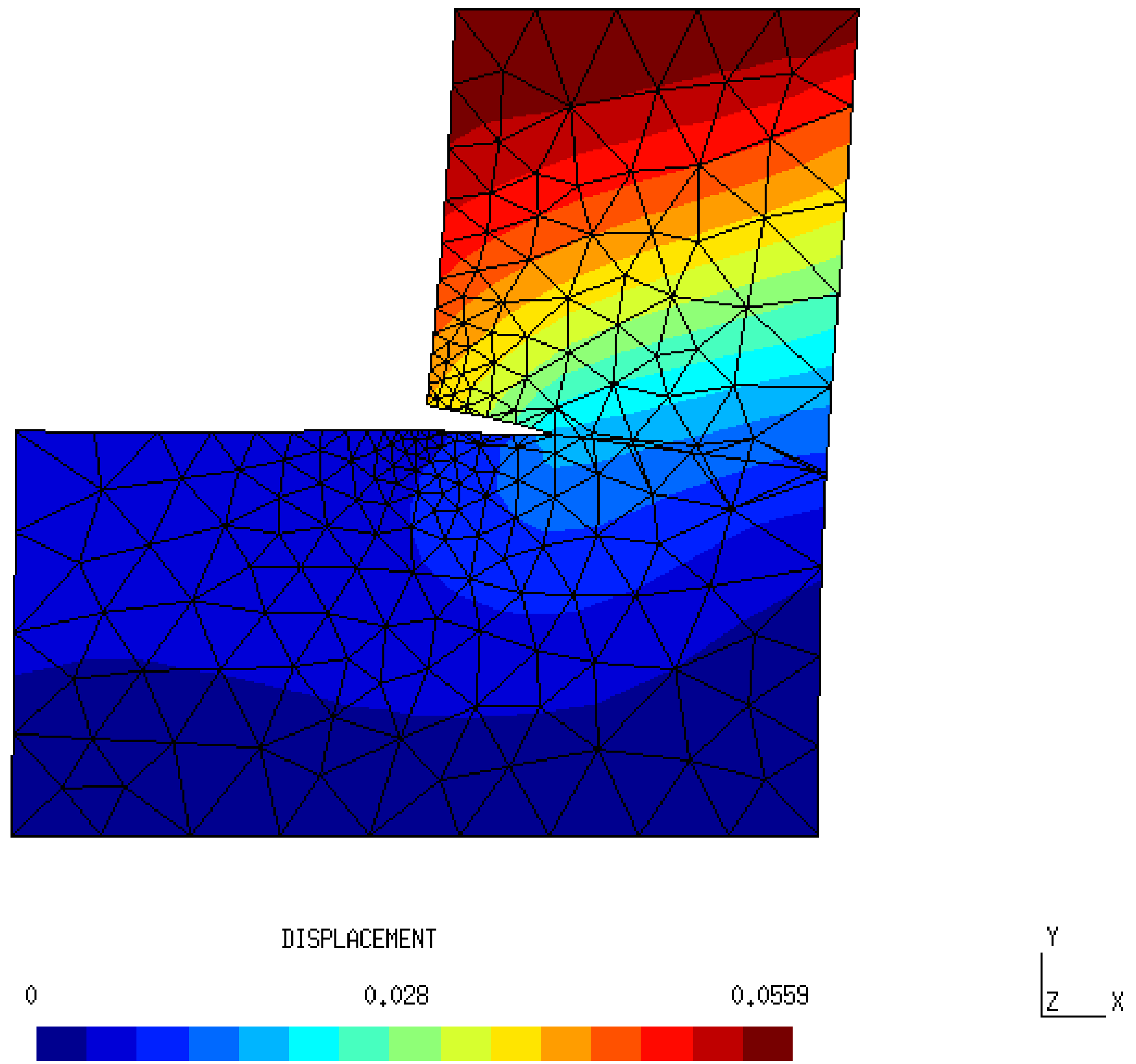
Extended Finite Elements

Propagation



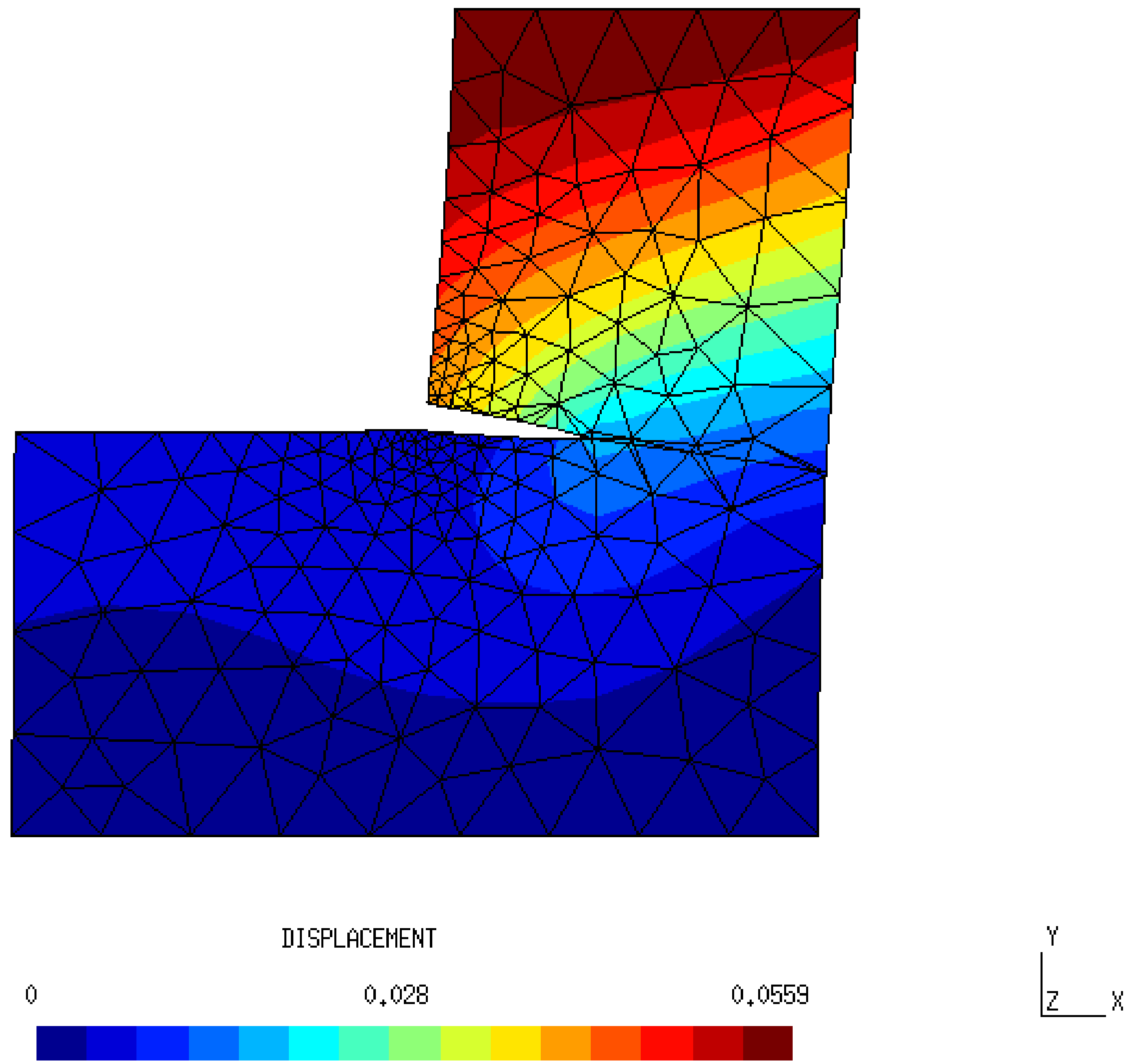
Extended Finite Elements

Propagation



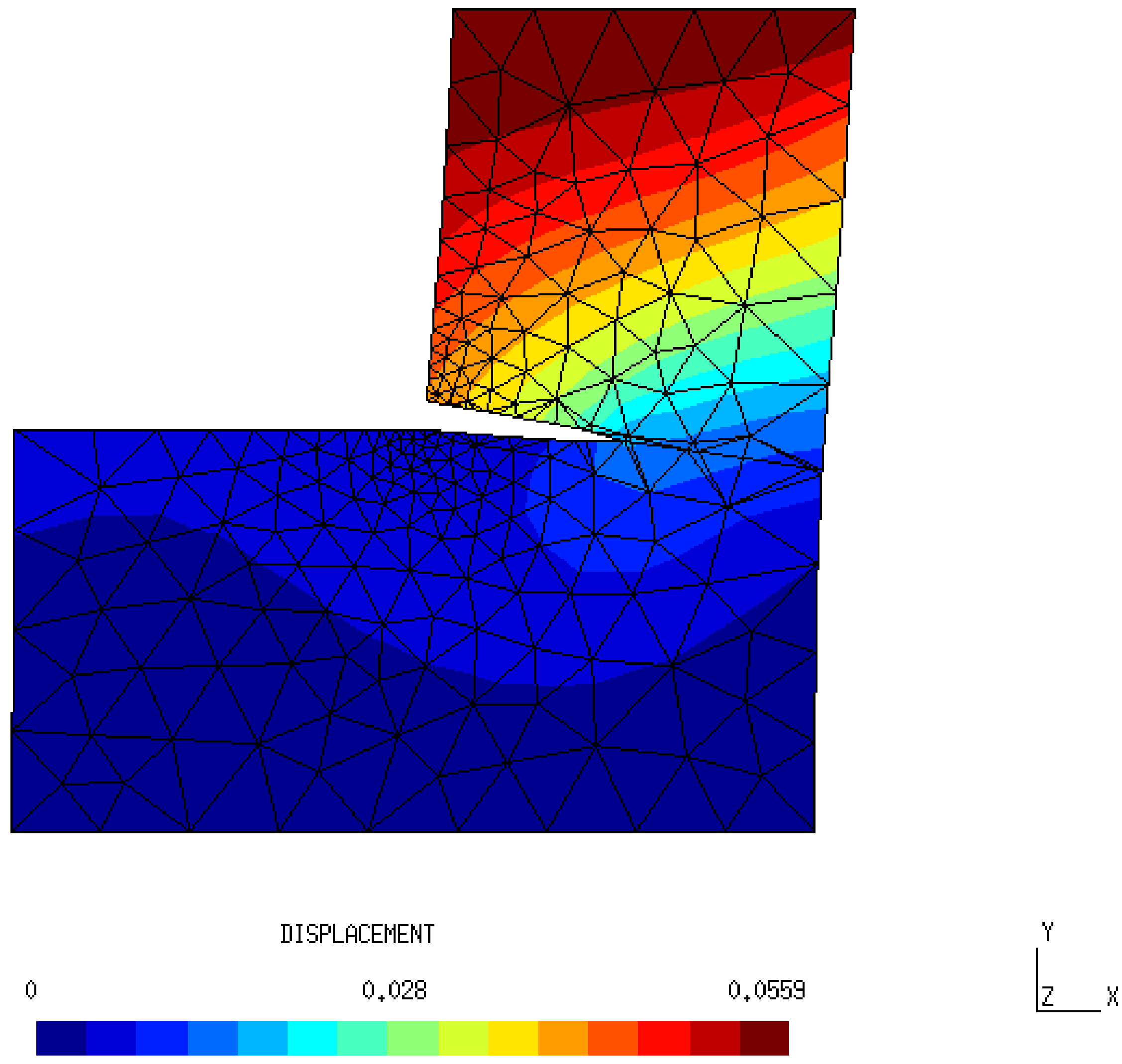
Extended Finite Elements

Propagation



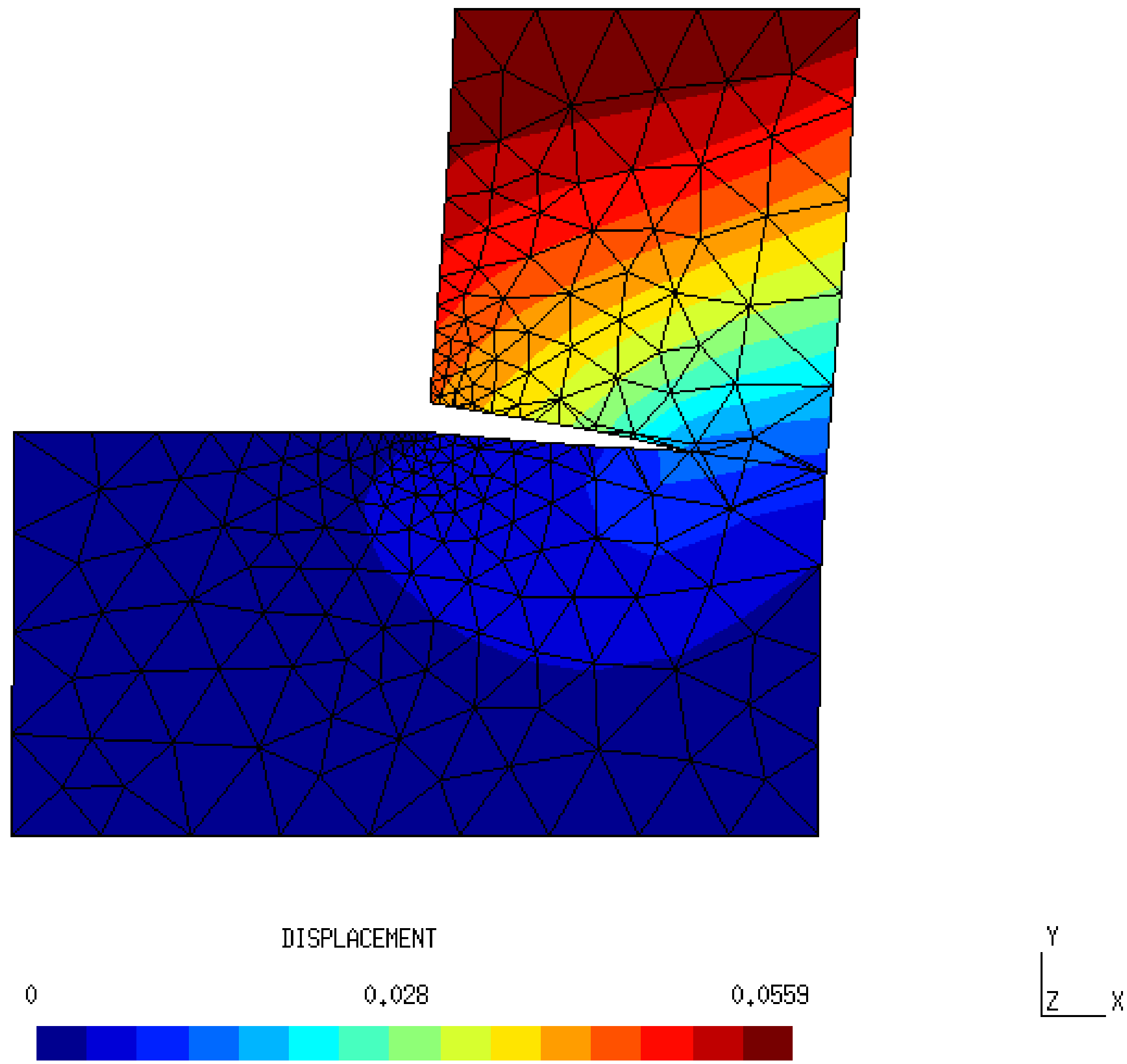
Extended Finite Elements

Propagation



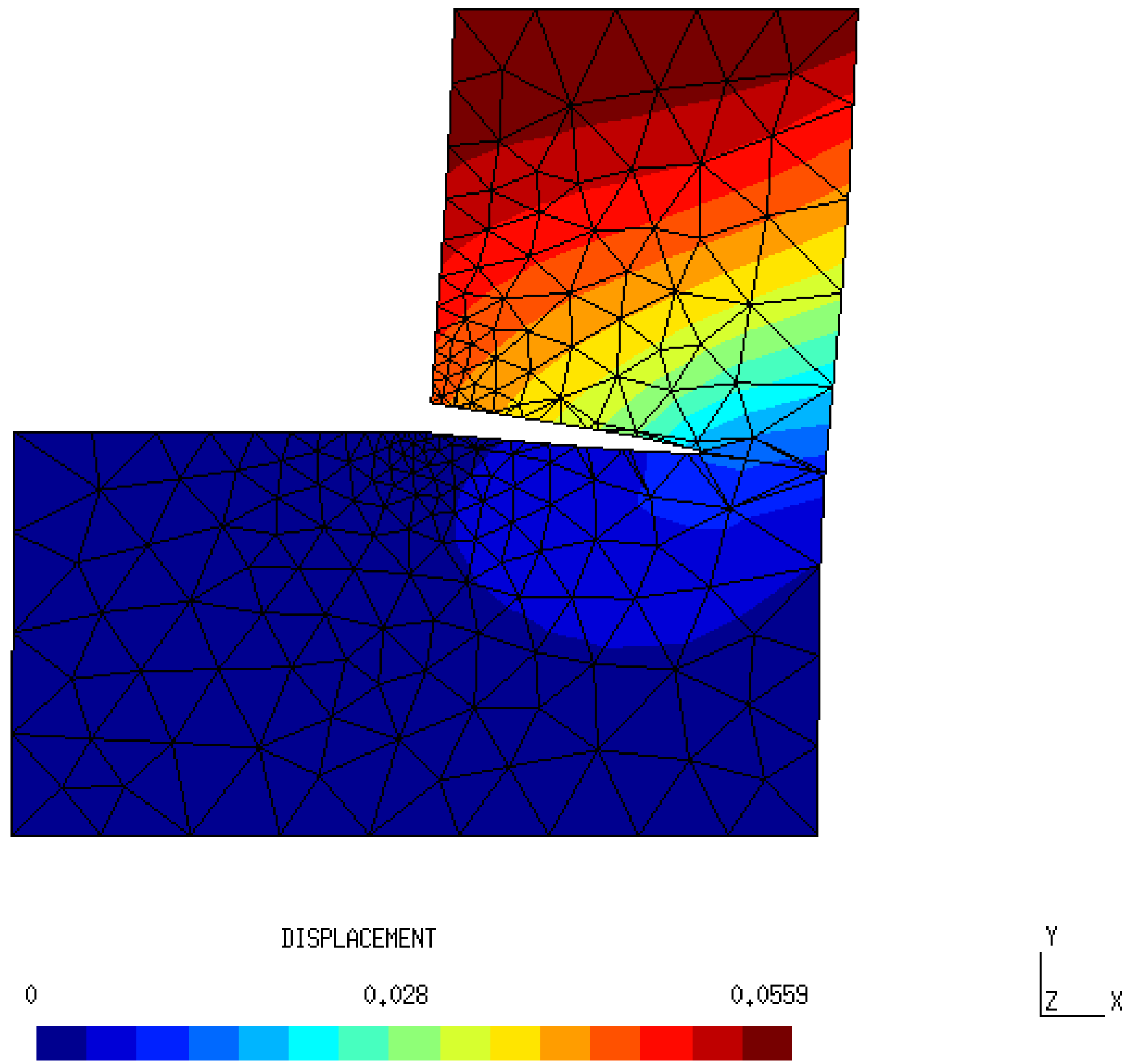
Extended Finite Elements

Propagation

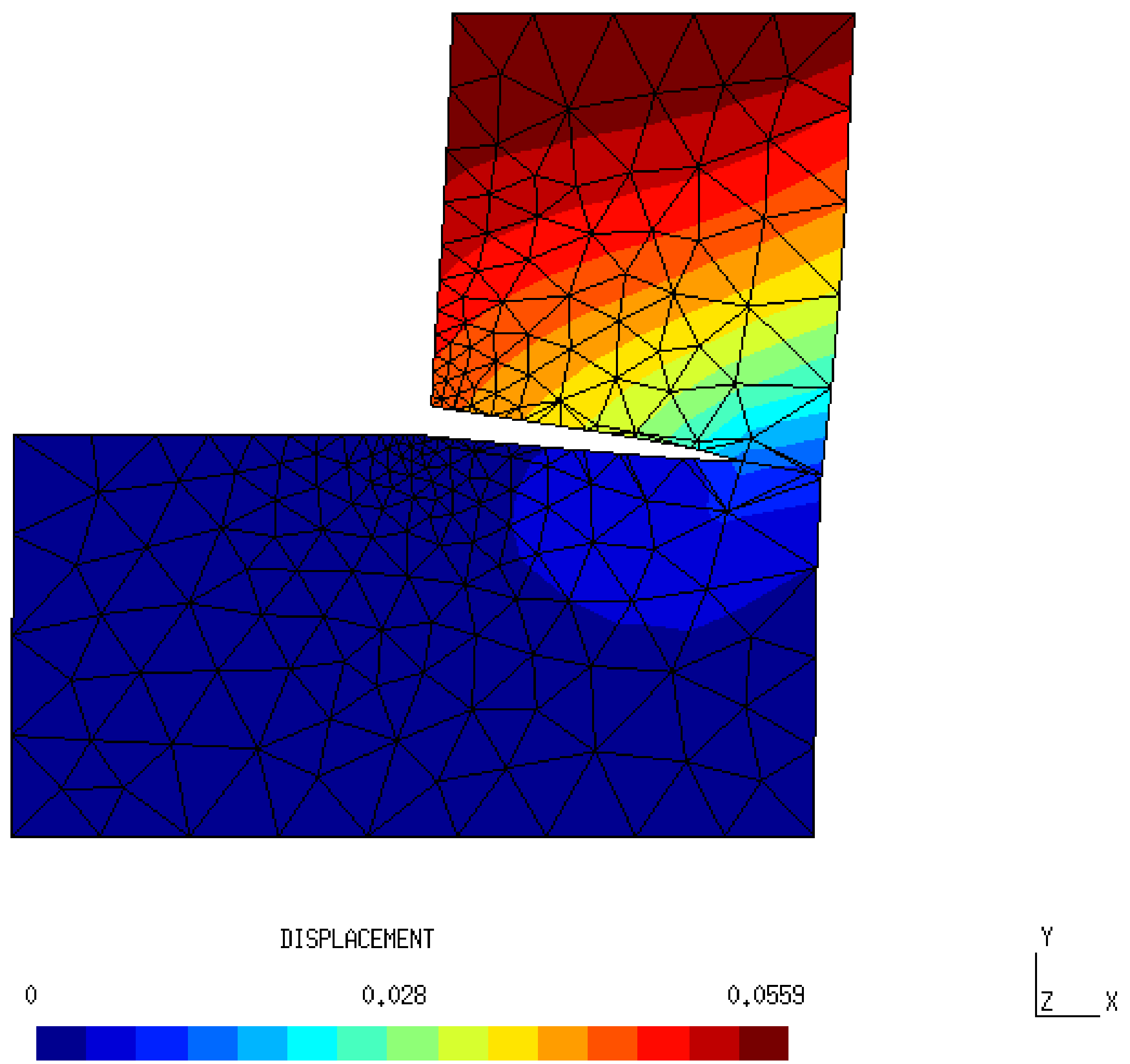


Extended Finite Elements

Propagation

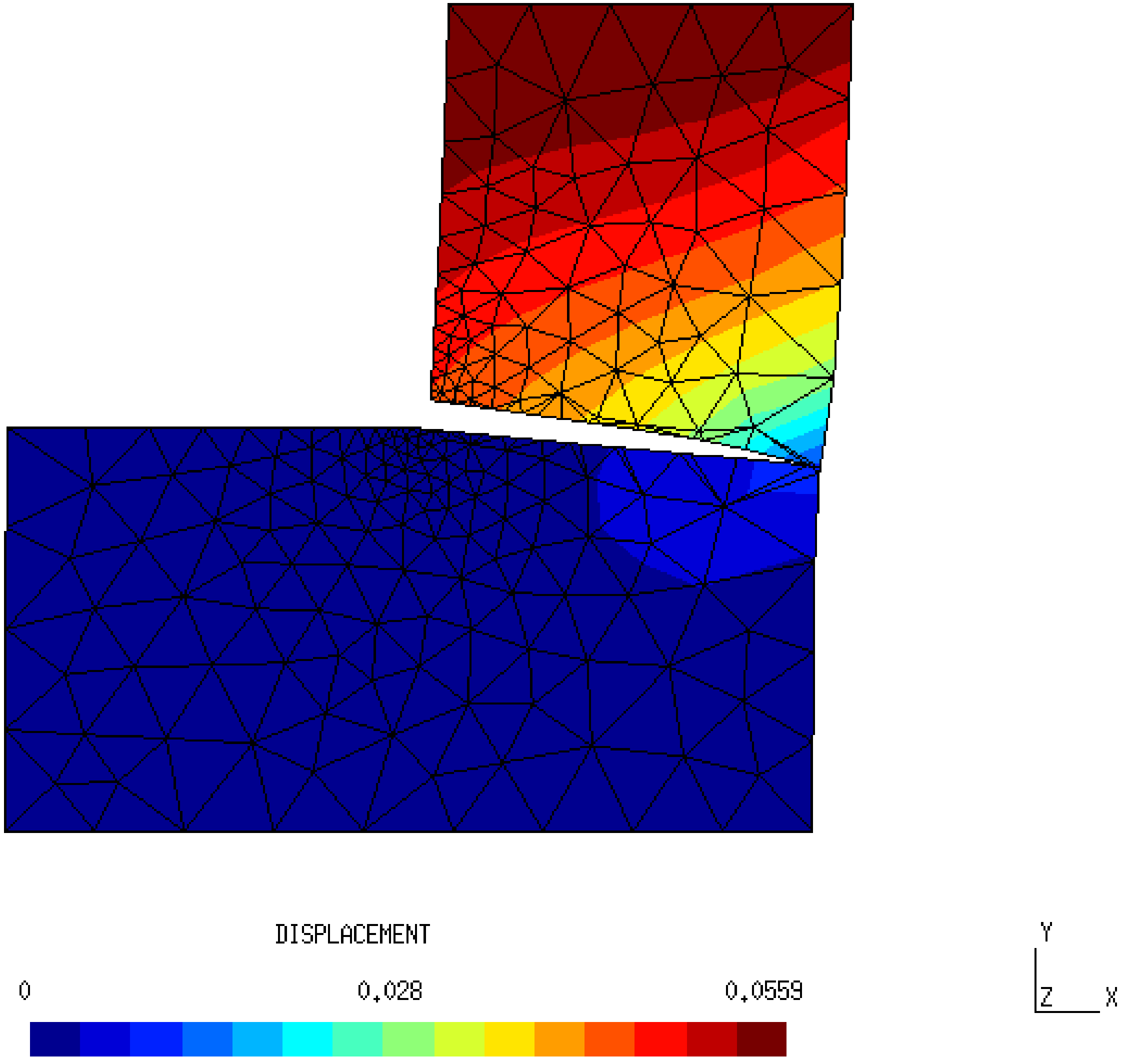


Propagation



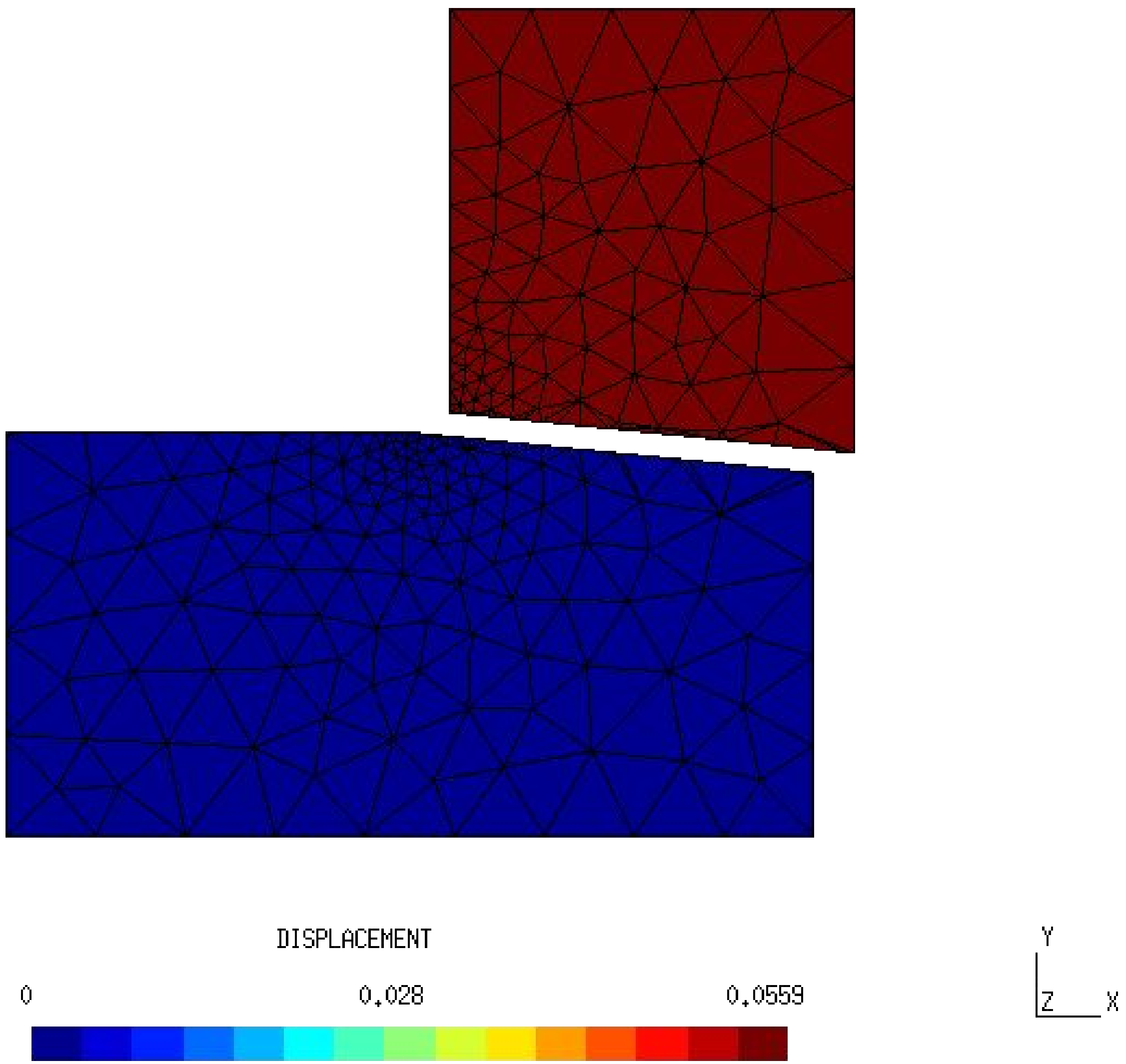
Extended Finite Elements

Propagation



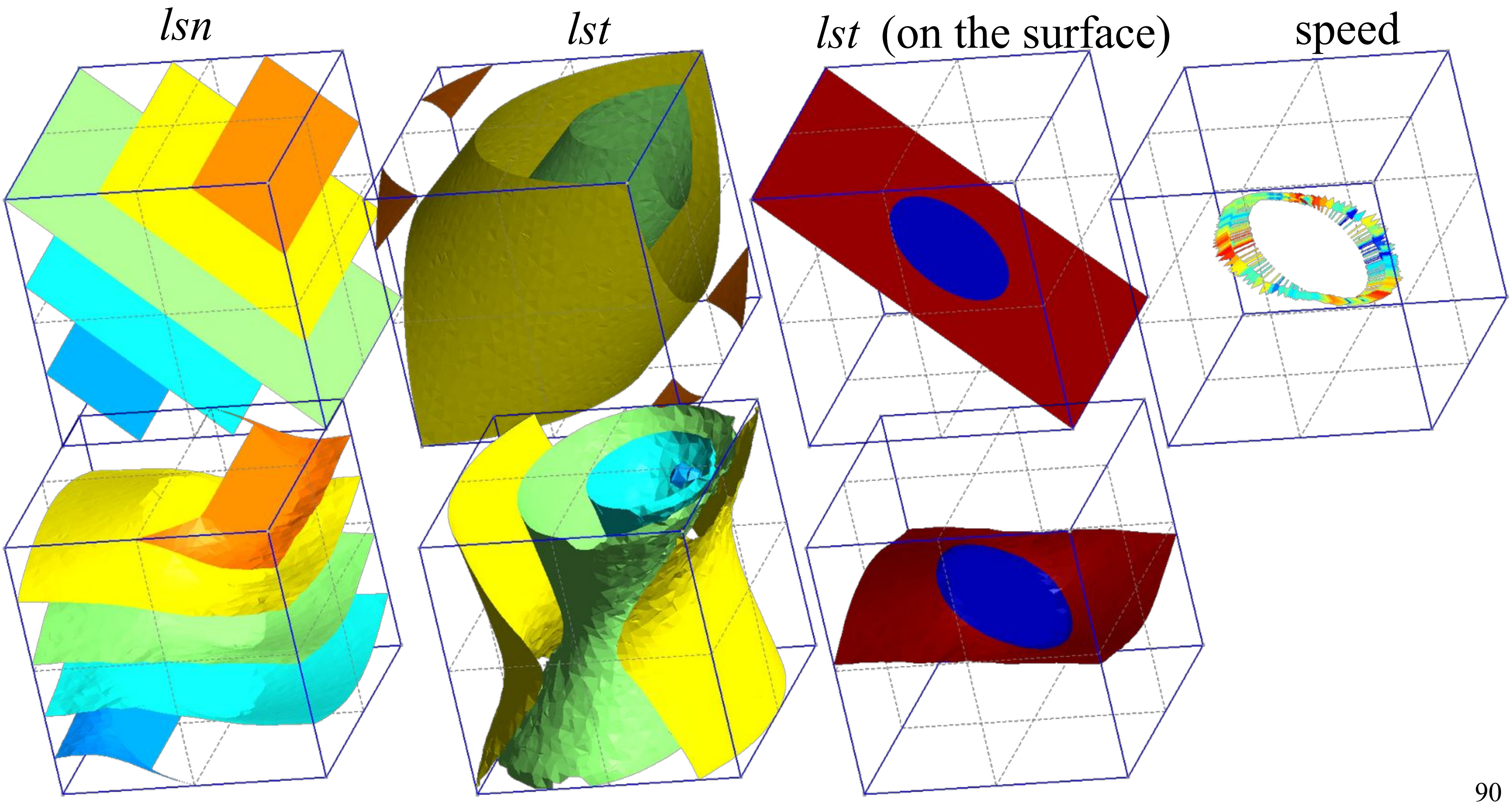
Extended Finite Elements

Propagation



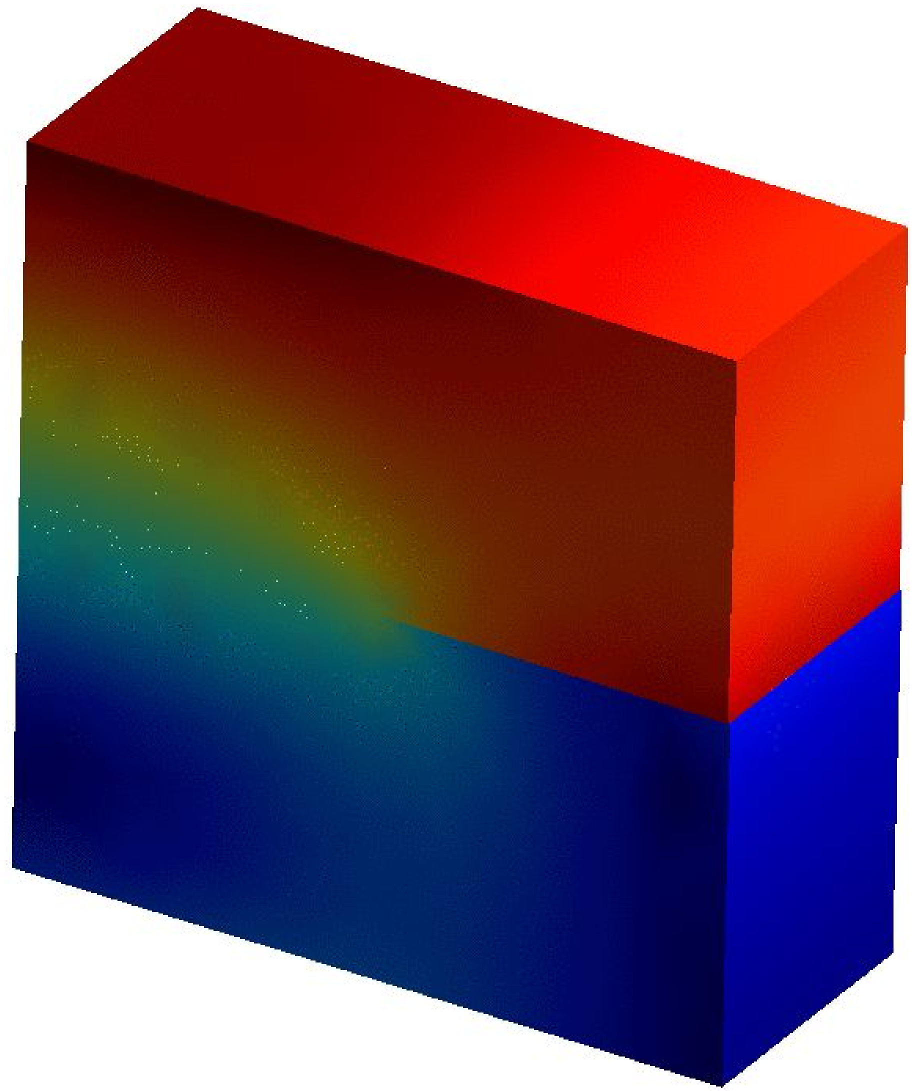
Extended Finite Elements

3D Propagation



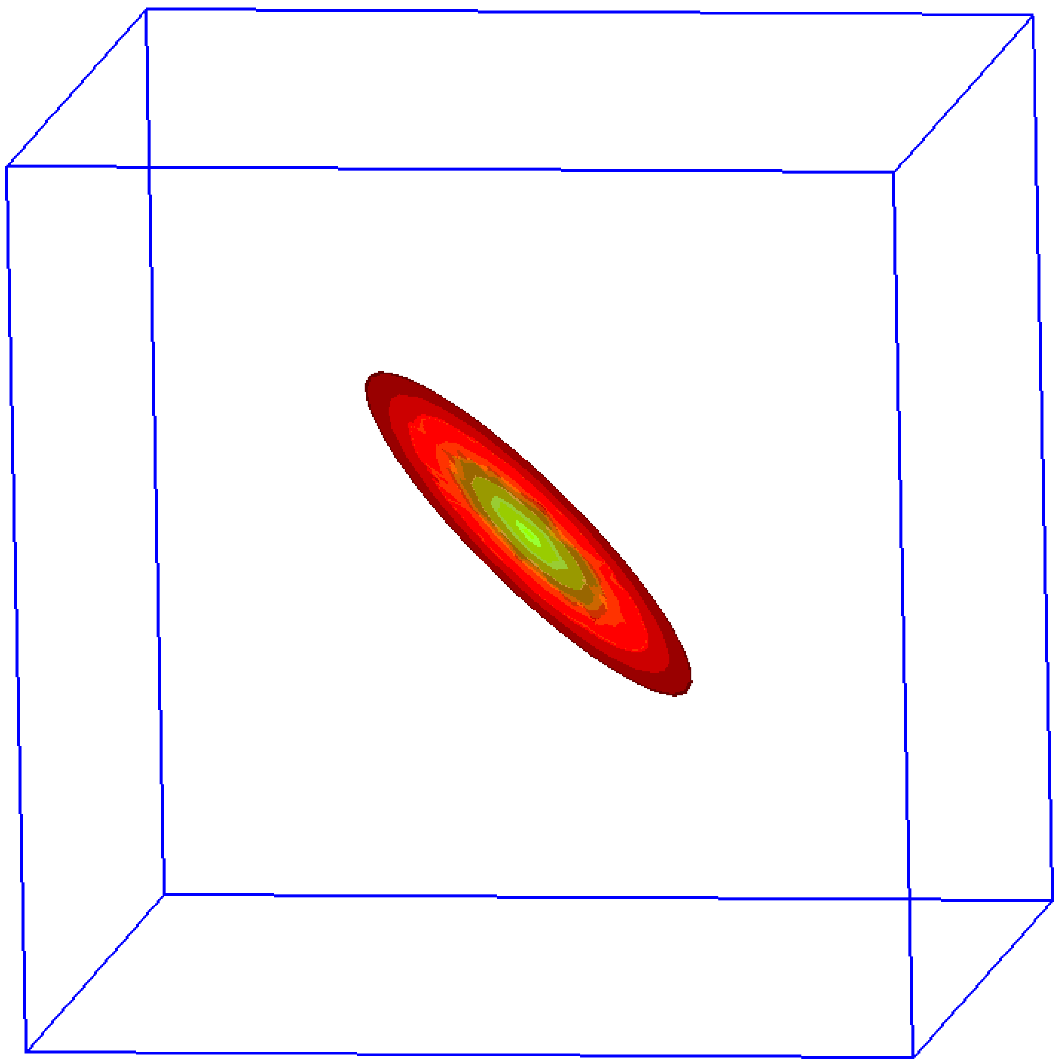
Extended Finite Elements

3D Propagation



Extended Finite Elements

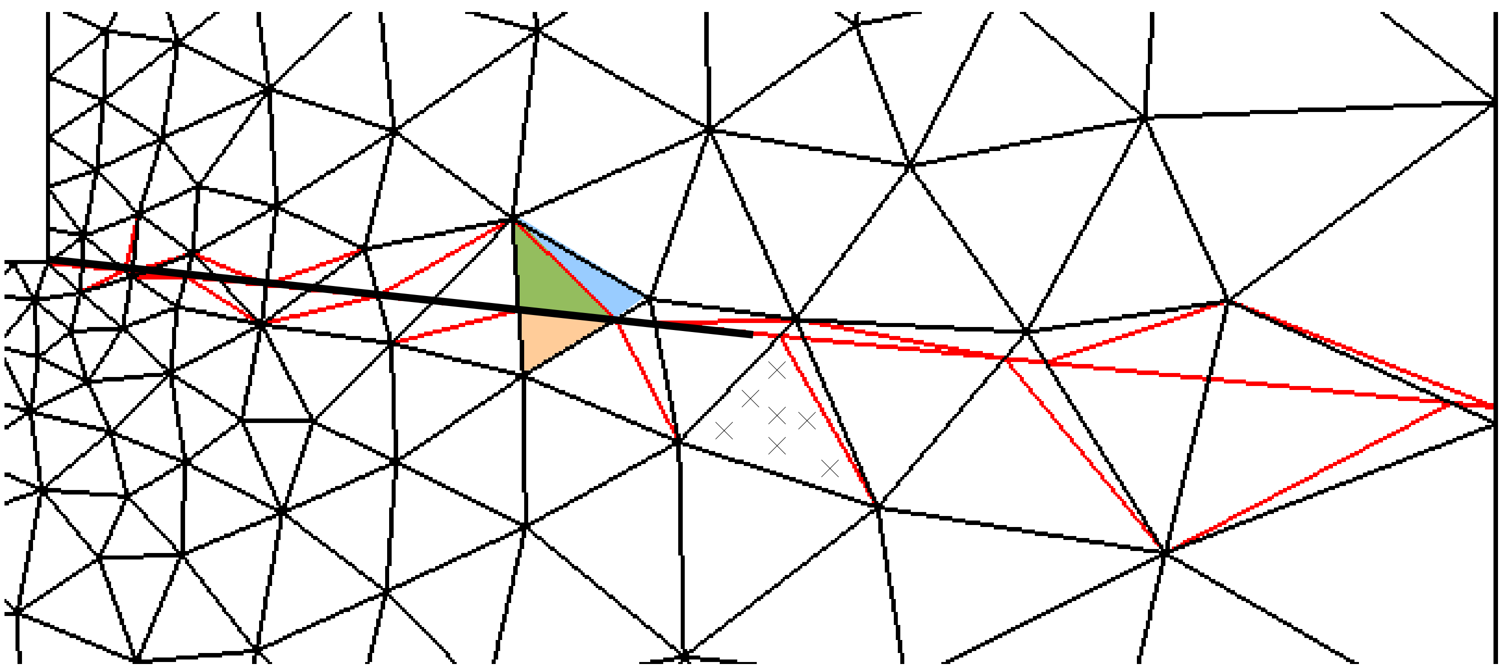
3D Propagation



Extended Finite Elements

Tricky points

- Integration
  - One should cut elements along the interface… but one should also change the quadrature or increase the number of quadrature points because the integrand is no more polynomial

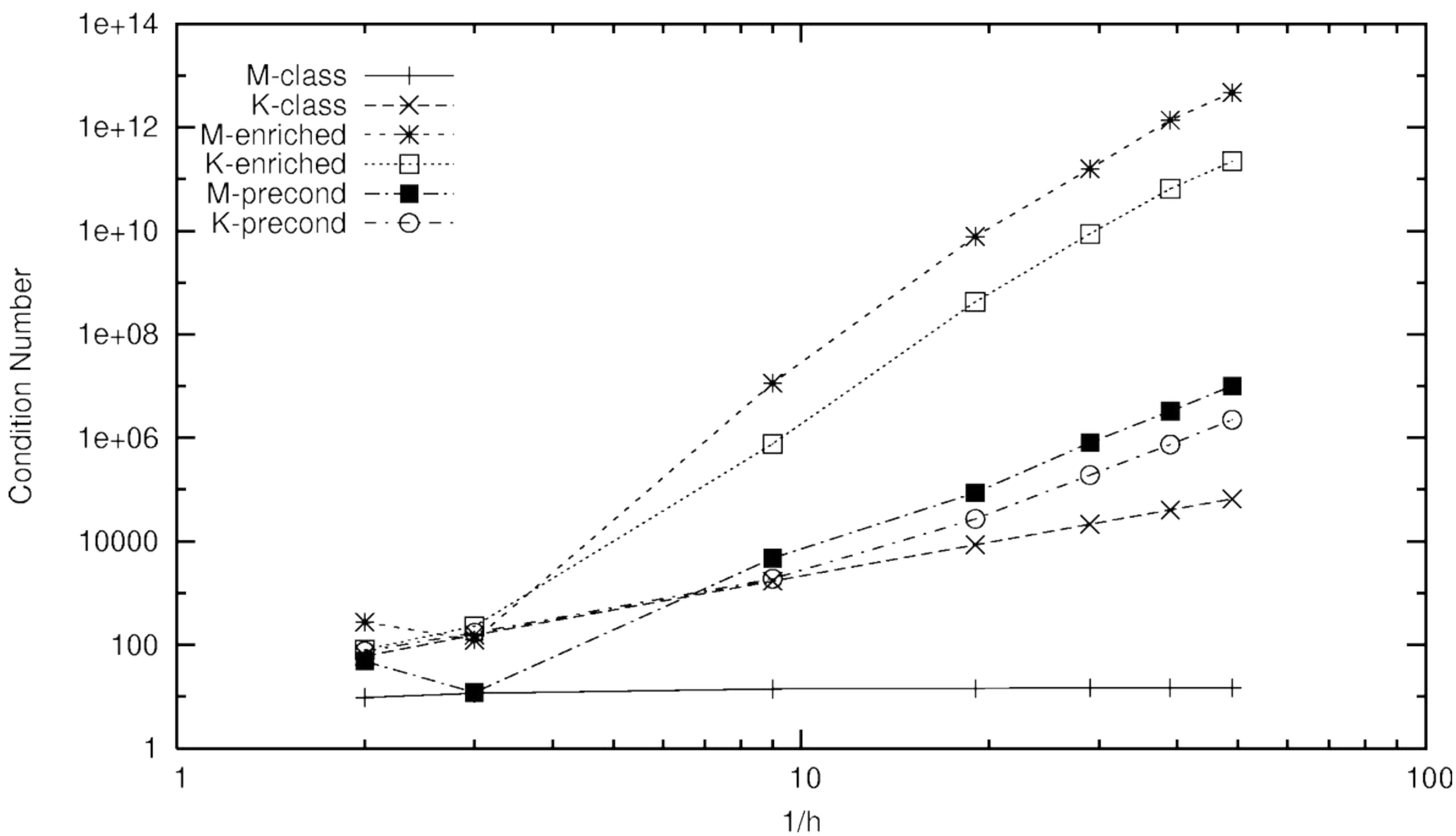


Tricky points

- Condition number
  - If the choice of the enriched DoFs is wrongly made, then the condition number will be close to 0 (this yields a singular linear system)
    - If the crack goes close to a node $\rightarrow$ then it goes through it (at least virtually)
  - The enriched shape functions at crack tip may induce a bad condition number (they “look alike”)
    - Use of a specialized preconditionner

Tricky points

- Condition number



Tricky points

- Representation valid for cracks in relatively fragile materials
  - The crack is represented using a infinitely thin line
  - The crack tip is a mathematical point, with non bounded values for the various mechanical fields
  - Propagation laws based on global energy-based laws (e.g. energy release rate G...)
- It is too restrictive to lead to good results for ductile materials

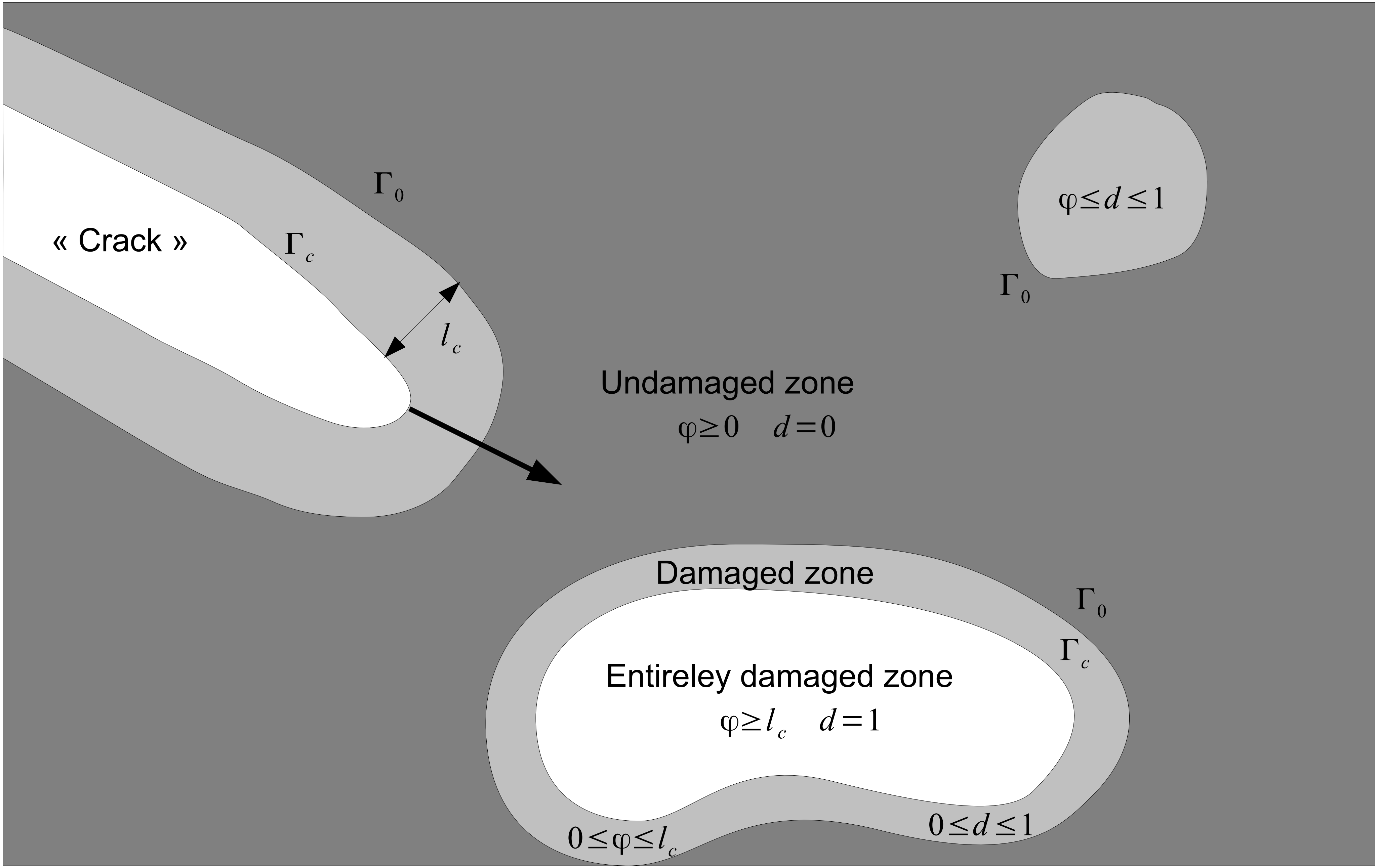
New properties

- Crack shape absolutely non trivial
- The propagation is made via a damage variable
- The level sets are used to represent at the same time

  - the damage variable d
  - the crack front (where d=1)
  - the boundary between the damaged zone (d>0) and the rest of the domain where the behavior is elastic
- Notion of “Thick” Level Set

Extended Finite Elements

Thick Level Set



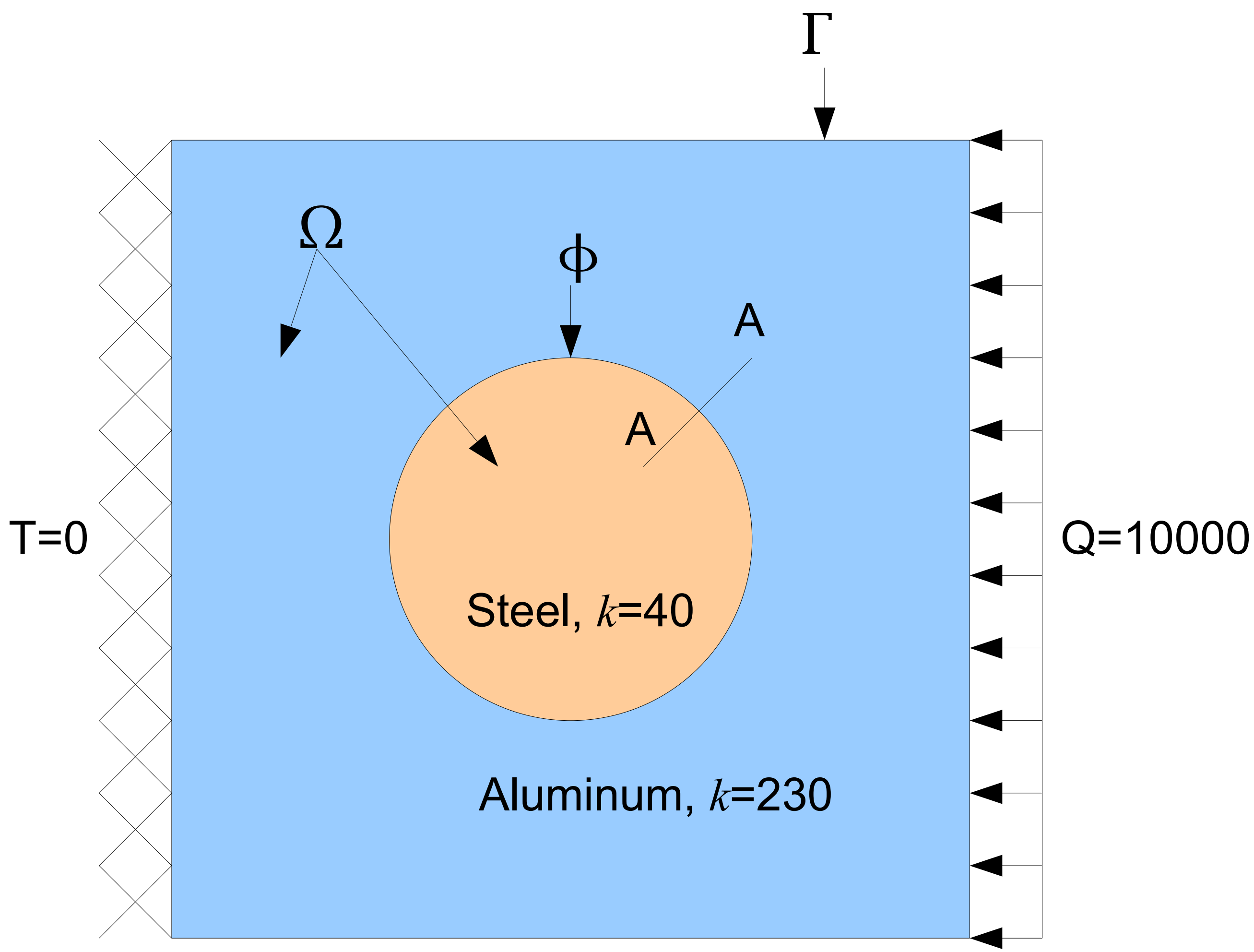
Extended Finite Elements

Thick Level Set

N. Moës, C. Stolz, P.-E. Bernard, and N. Chevaugeon
A level set based model for damage growth: The thick level set Approach Int. J. Numer. Meth. Engng 2011; 86:358–380 DOI: 10.1002/nme.3069

Problems with a jump in the gradient ("dual" variable)

- Thermal transfer model problem



Bi-material interface

- The interface is represented by the following level-set :

  $$\phi = \{ x \in \Omega \ / \ ls(x) = 0 \}$$

- This interface can be of complex geometrical shape and / or changing in time
  - Again, no mesh conformity

- Finite element model (again, homogeneous boundary conditions)

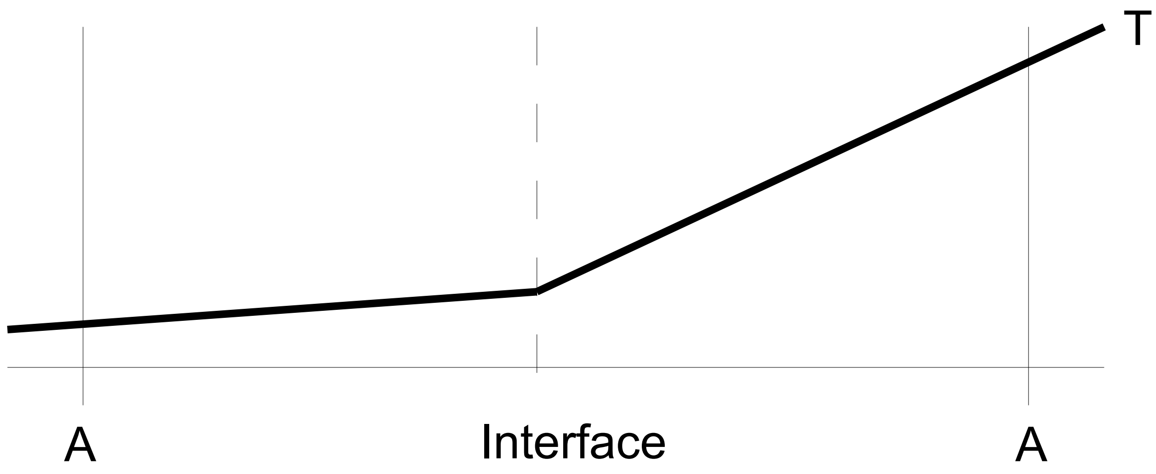
$$\text{Find } u \in H_0^1(\Omega) \text{ s.t.}$$
$$a(u, v) = b(v) \quad \forall v \in H_0^1(\Omega)$$

with

$$a(u, v) = \int_\Omega k(\nabla u \cdot \nabla v)\, d\Omega \qquad b(v) = \int_\Gamma f(x) \cdot v\, d\Gamma$$

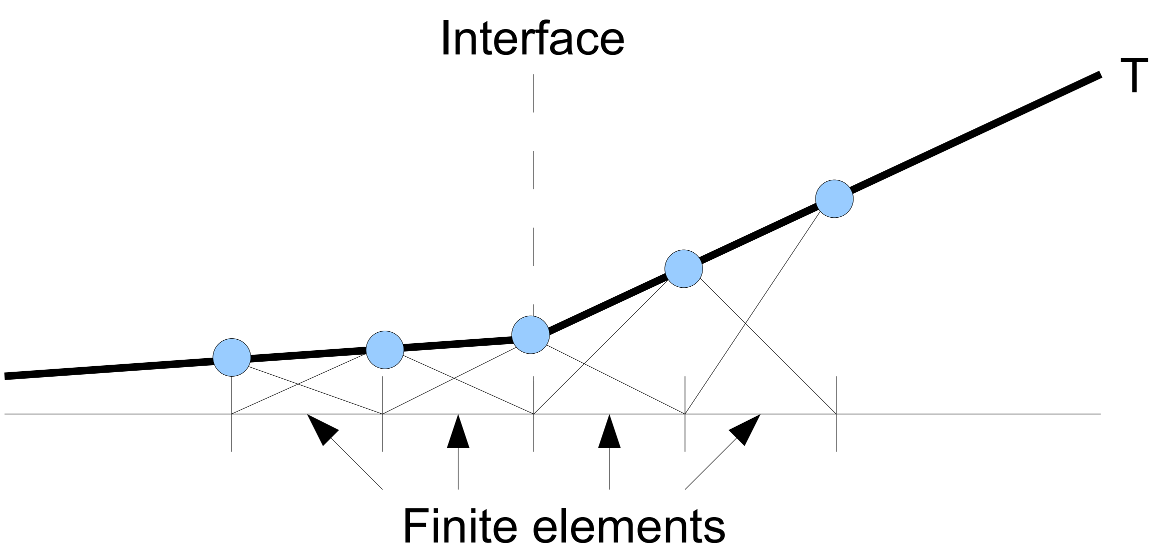
Bi-material interface

- We want to be able to represent the right temperature profile along the interface
  - A-A cut : Theoretical temperature profile



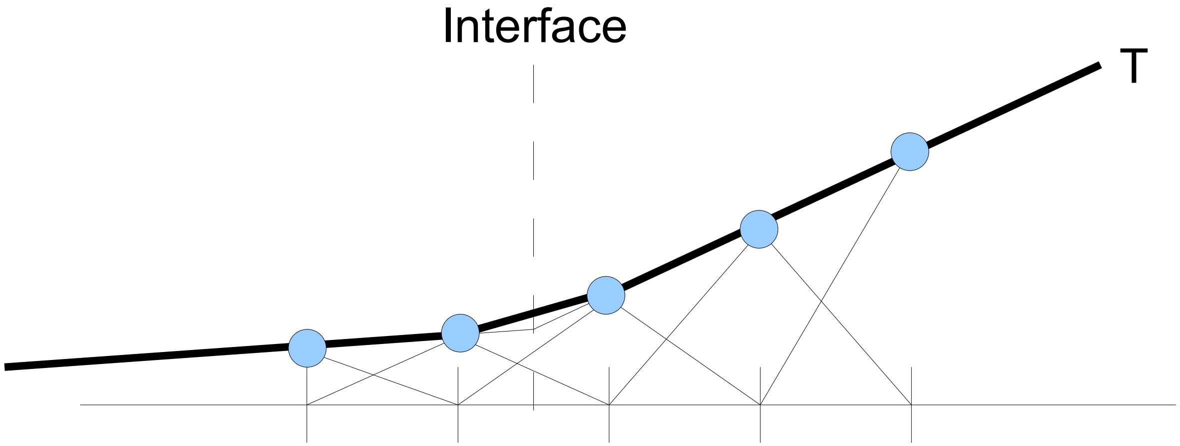
Bi-material interface

- The discontinuity is on the derivative of T
- If the interface is exactly on element boundaries, then the discontinuity is naturally belonging to the function space



Bi-material interface

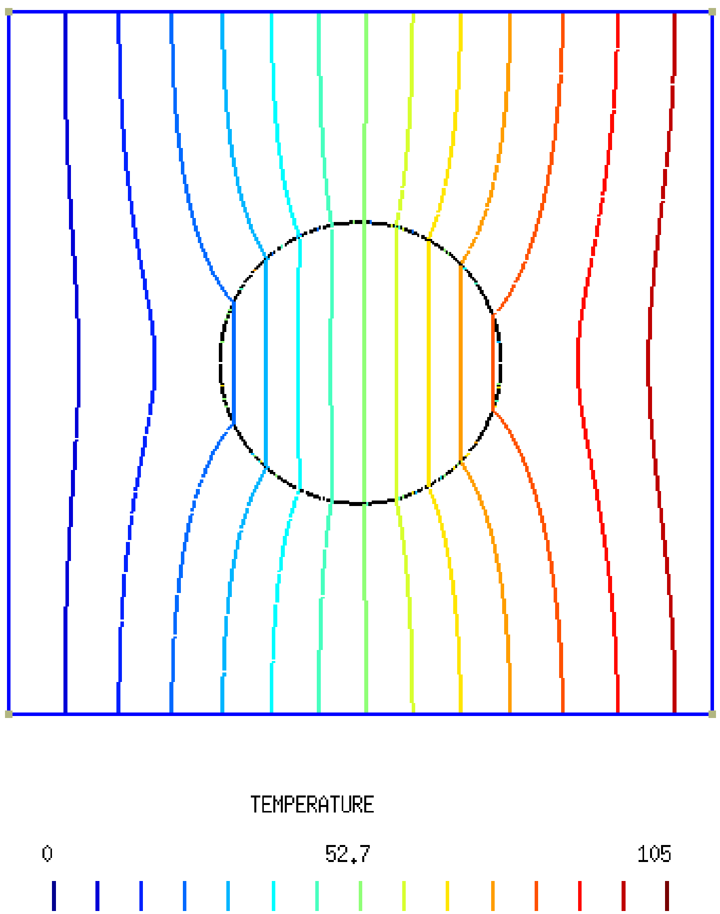
- The discontinuity is on the derivative of T
- If the interface is not exactly on element boundaries, then ...



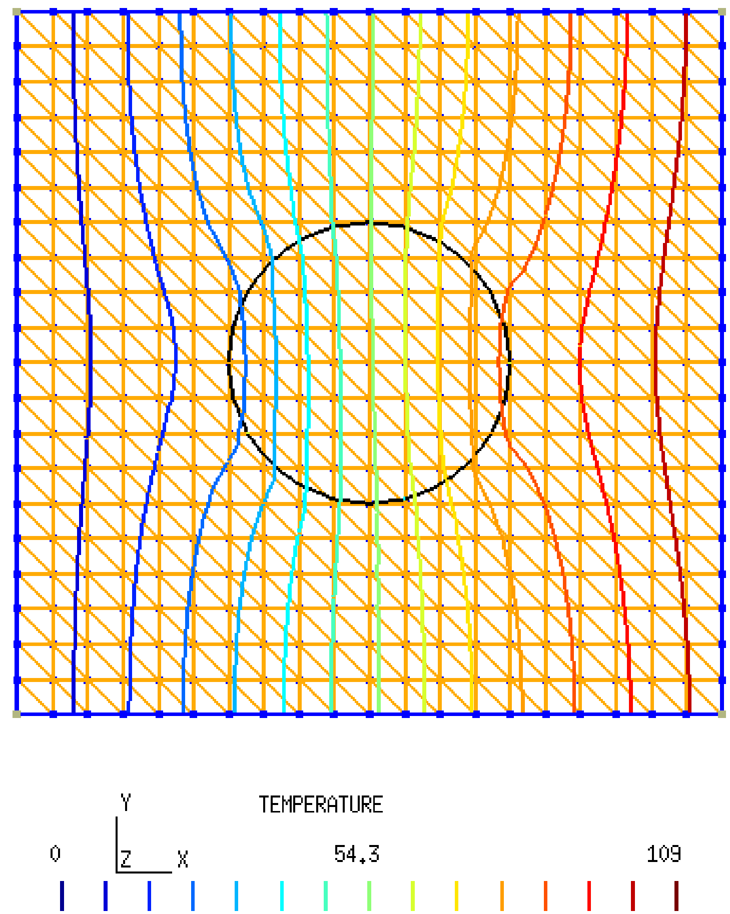
Bi-material interface

- This explains the very approximate solution...

Exact solution



Standard F.E. solution



Extended Finite Elements

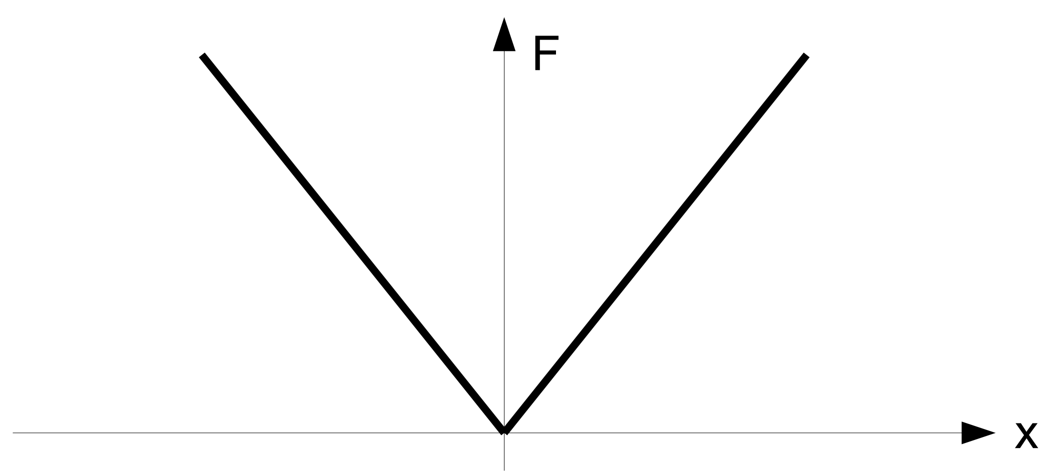
Bi-material interface

- The idea here is to enrich the function space so that the discontinuity (in the gradient) belong to it.

$$u(x)=\sum_{i\in\Omega}\lambda_i N_i(x)+\sum_{j\in C}\lambda_j^* N_j(x)\cdot F(x)$$

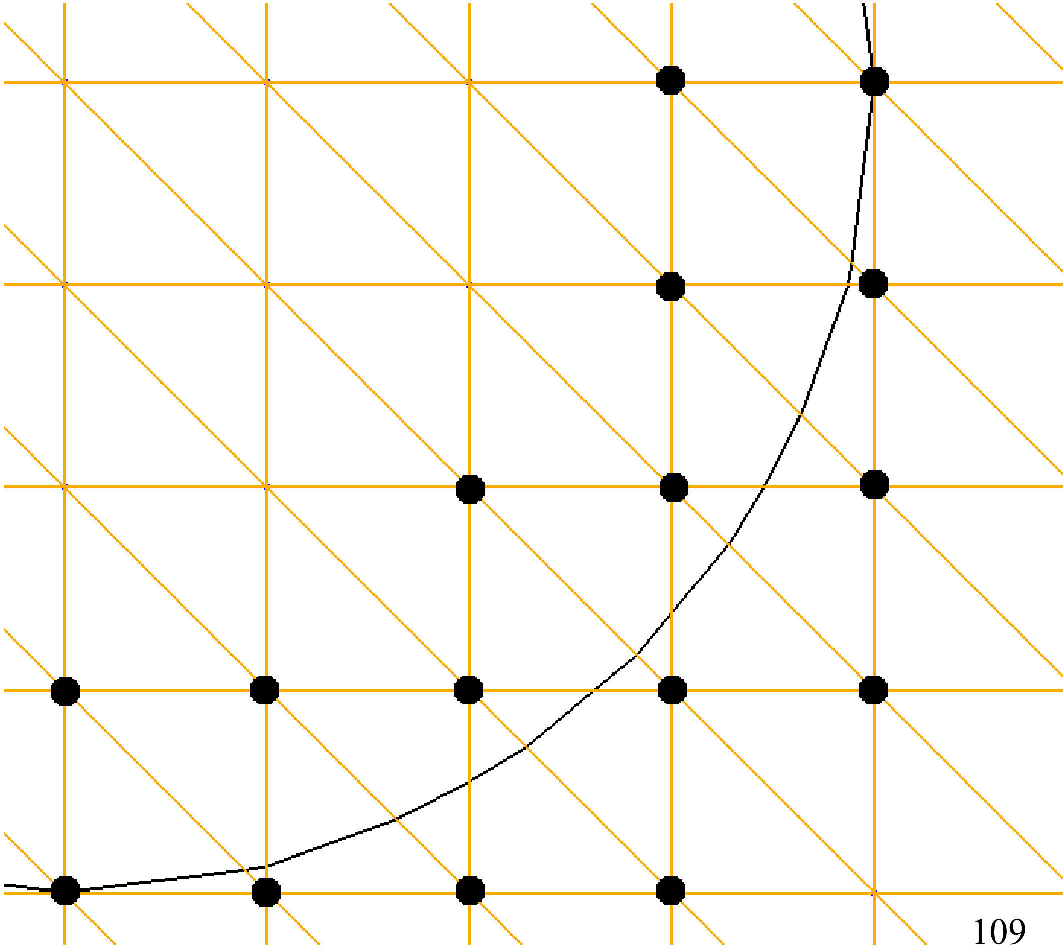
There exists many possibilities. One very simple is using directly the absolute value of the level-set.

$$F_1(x)=|ls(x)|$$



Bi-material interface

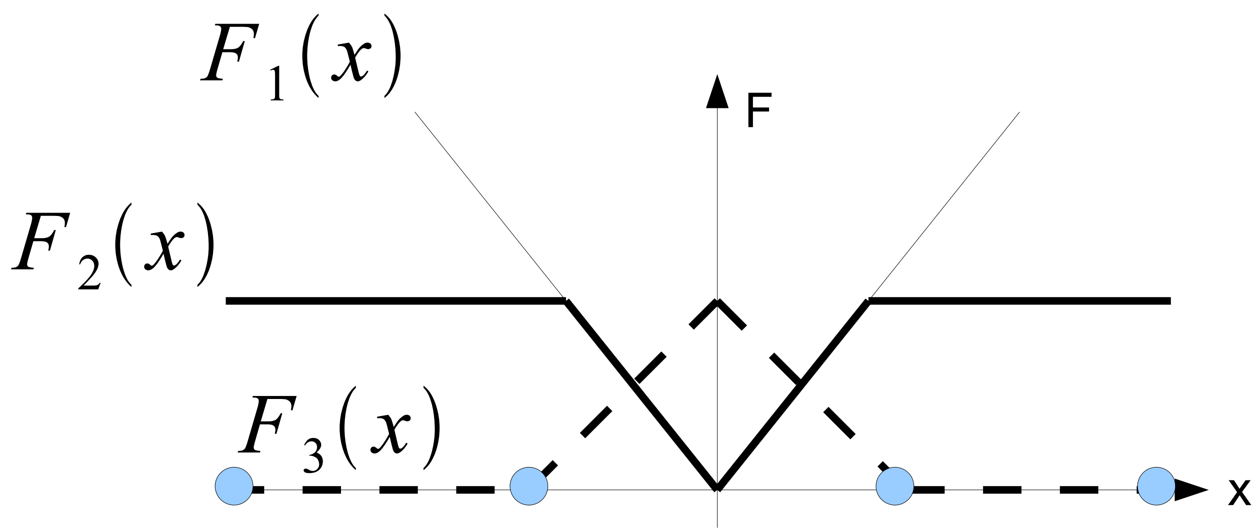
- Definition of the set C of the enriched nodes
  - This time, the nodes where at least one element of the support are cut must be enriched
  - In particular, if the interface is along edges, there is no enrichment



- Here are some other enrichment functions

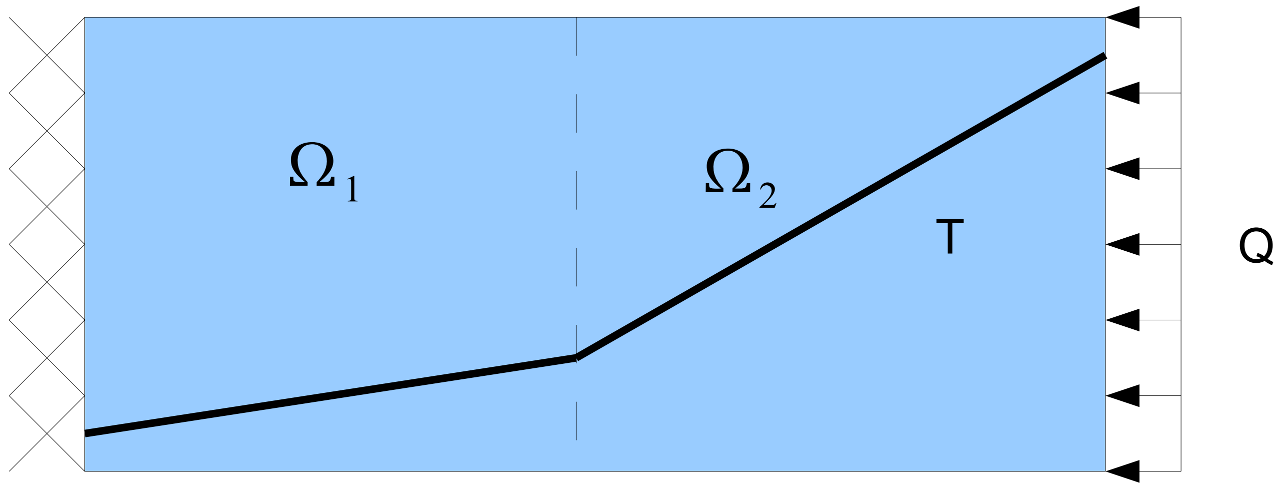
$$F_2(x) = \begin{cases} |ls(x)| \text{ in cut elements} \\ 1 \text{ else} \end{cases}$$

$$F_3(x) = \sum_i |ls_i| \cdot N_i(x) - \left|\sum_i ls_i \cdot N_i(x)\right|$$



Bi-material interface

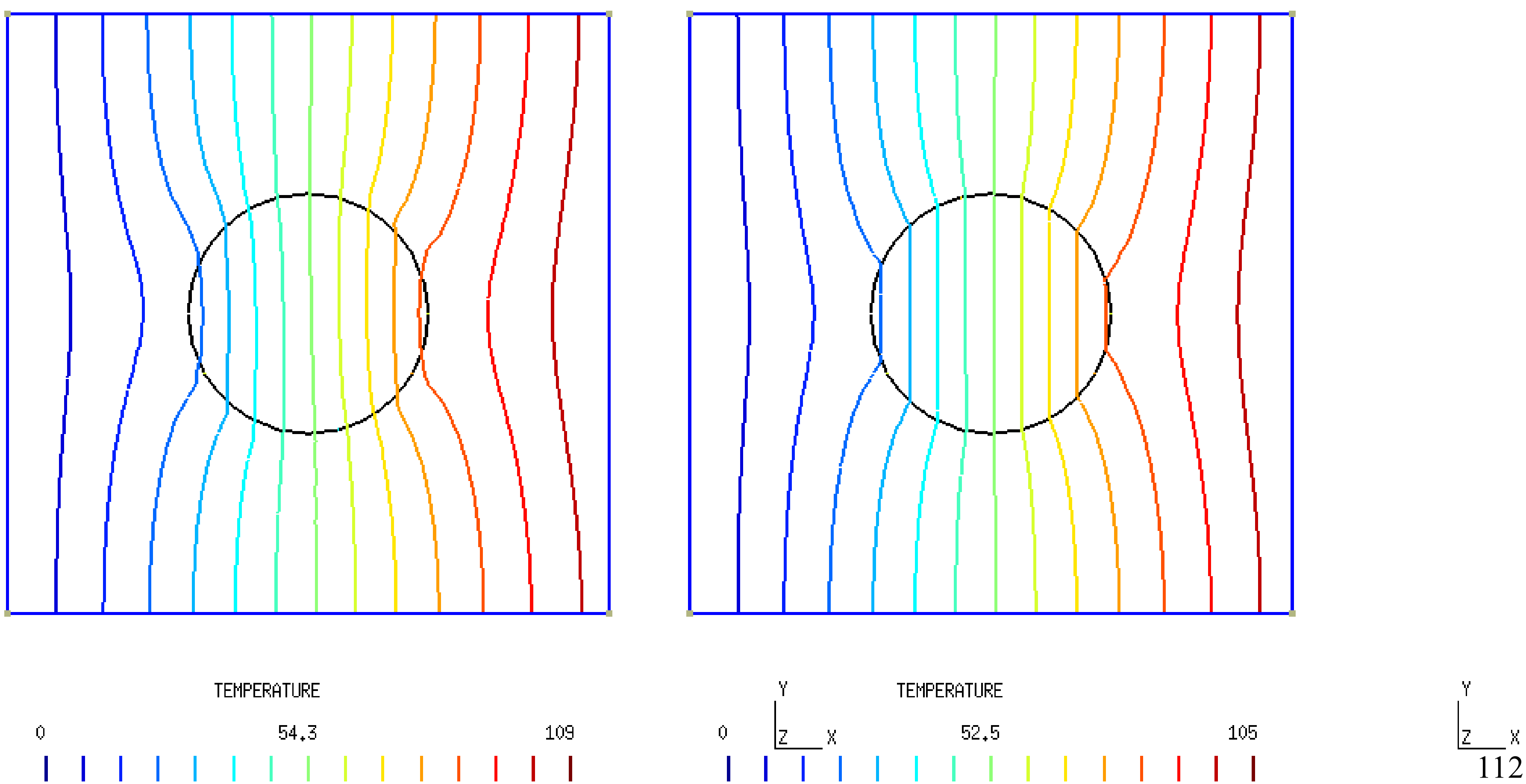
- Practically speaking, $F_3(x)$ gives the best results
  - On a simple model problem, the functions $F_2(x)$ and $F_3(x)$ are unable to give back the exact solution (which is linear by parts) when the interface does not belong to the mesh, but $F_1(x)$ does.



Bi-material interface

- Comparison of the solution with the right enrichment function

Solution without enrichment | Solution with enrichment

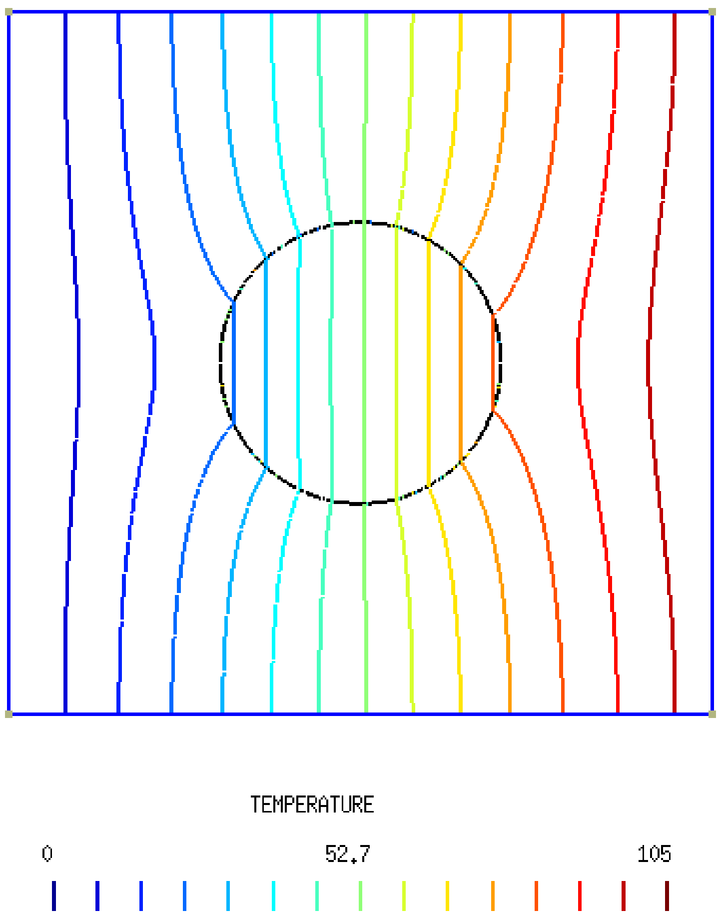


Extended Finite Elements

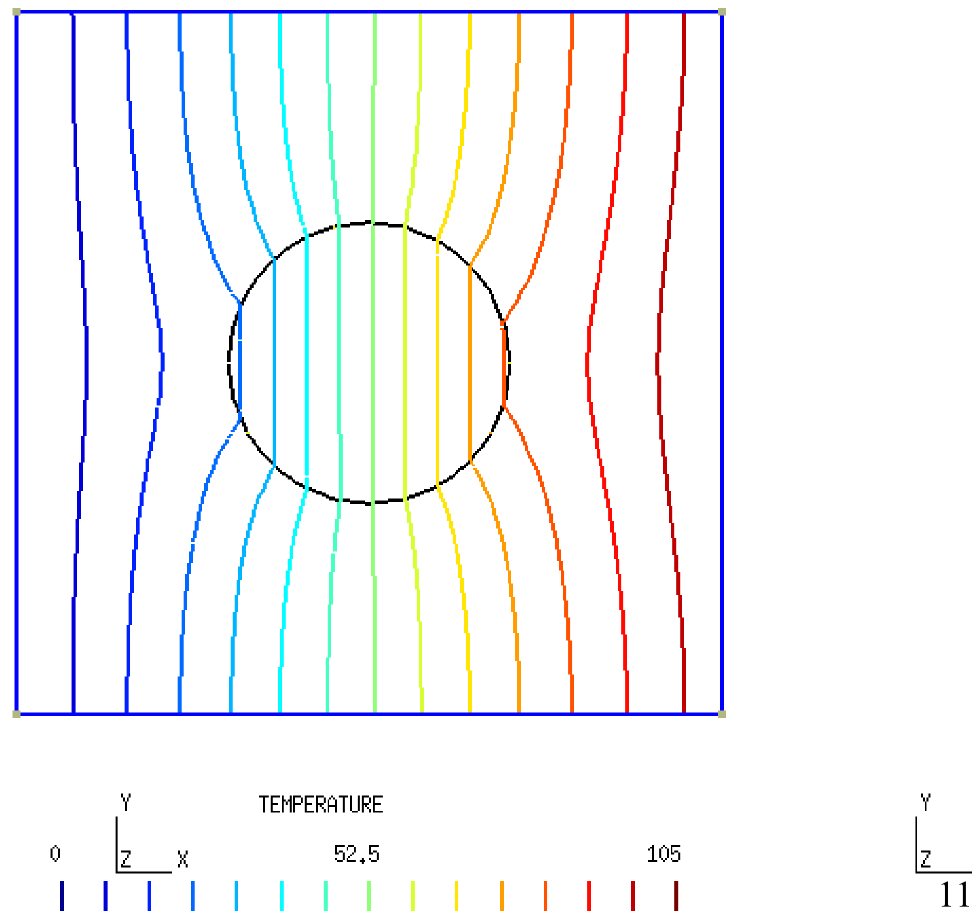
Bi-material interface

- Comparison of the solution with the right enrichment function

Exact solution

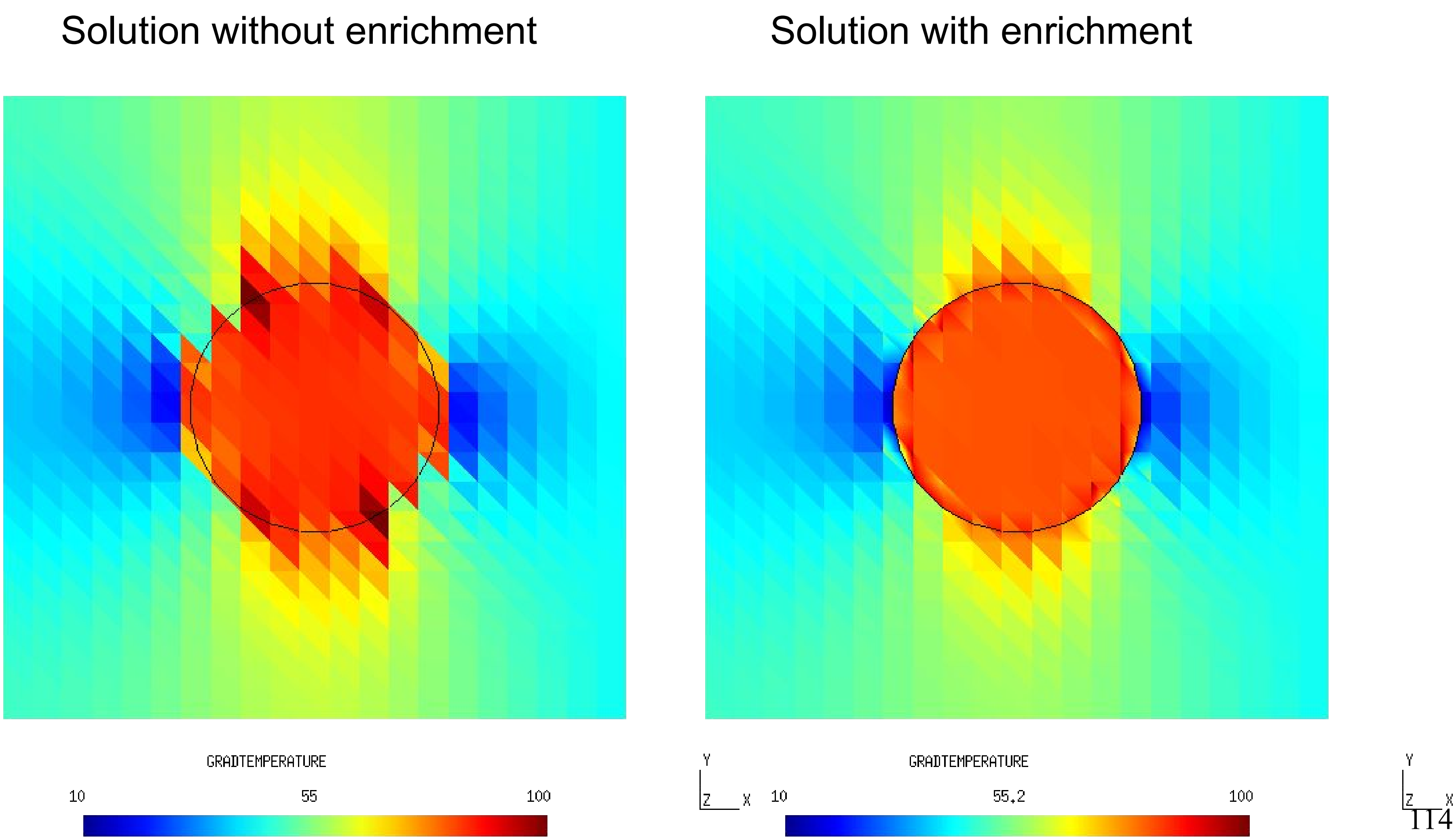


Solution with enrichment



Bi-material interface

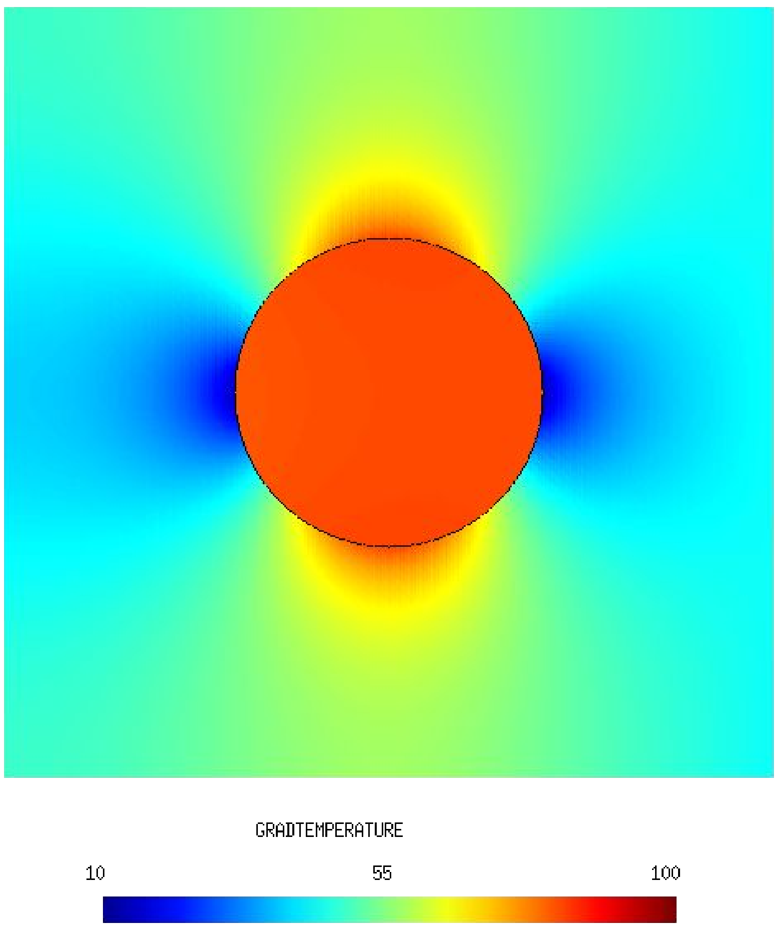
- Comparison of the gradient



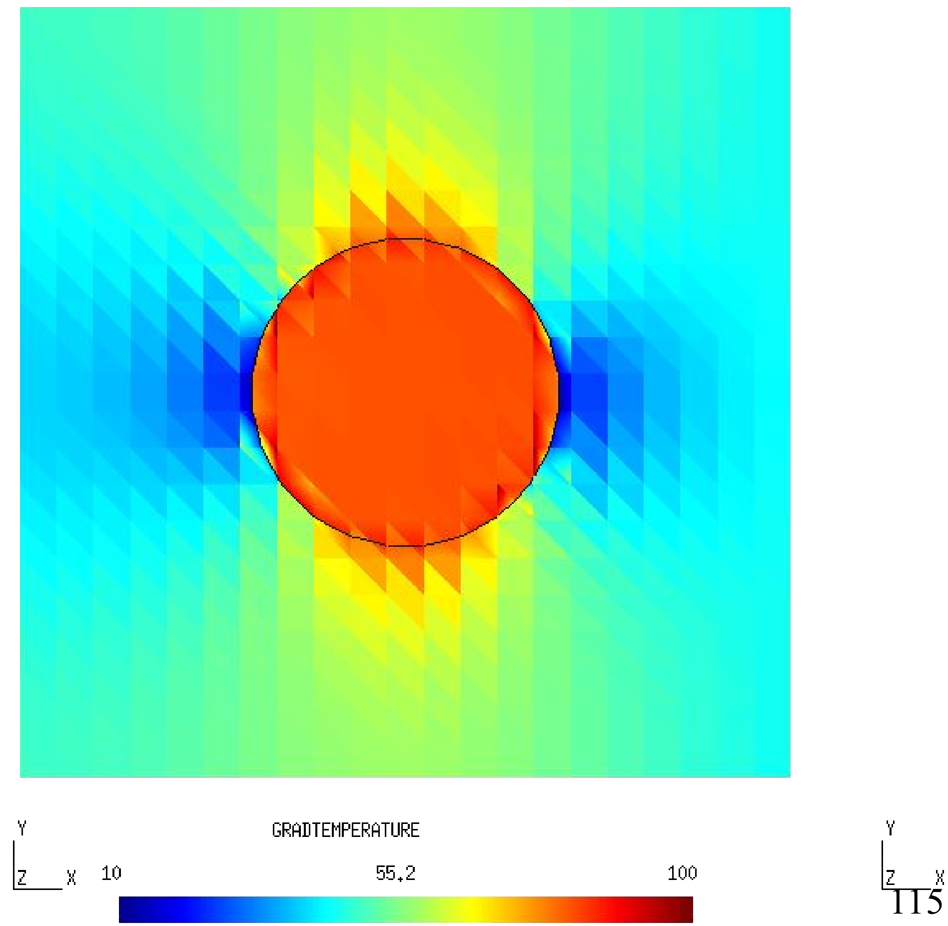
Bi-material interface

- Comparaison du gradient

Exact solution

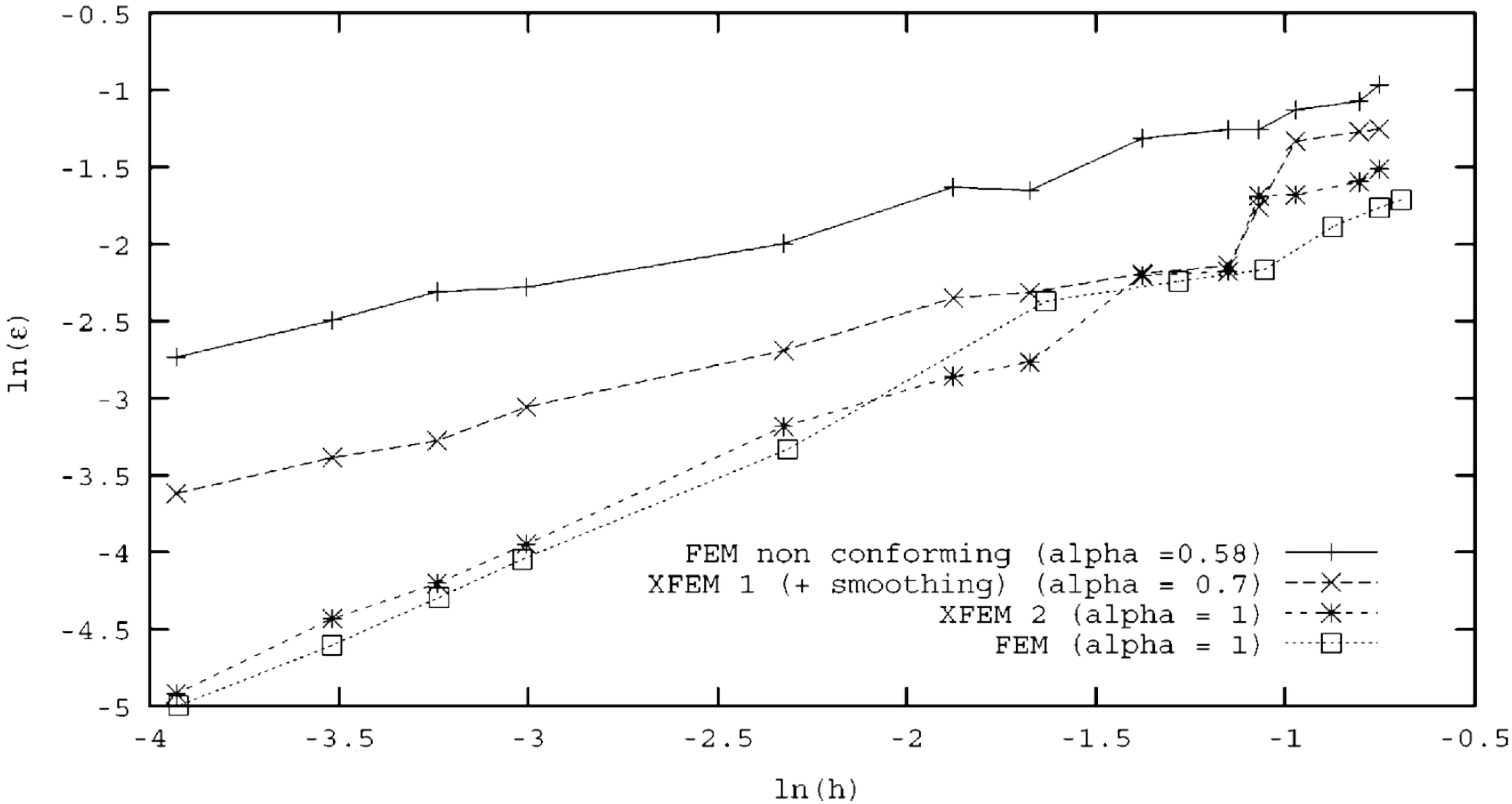


Solution with enrichment



Bi-material interface

- Convergence



More applications

- Discontinuities in the primal variable
  - Cracks
    - non linearities, plasticity
    - Dynamic effects (fast propagation)
  - Solidification front propagation
    - hydrogels
  - Discontinuities in the derivatives
    - Homogeneization

More applications

- Applications to other materials
  - Confined plasticity
  - Composites materials
  - Piezoelectric materials
  - Etc...

More applications

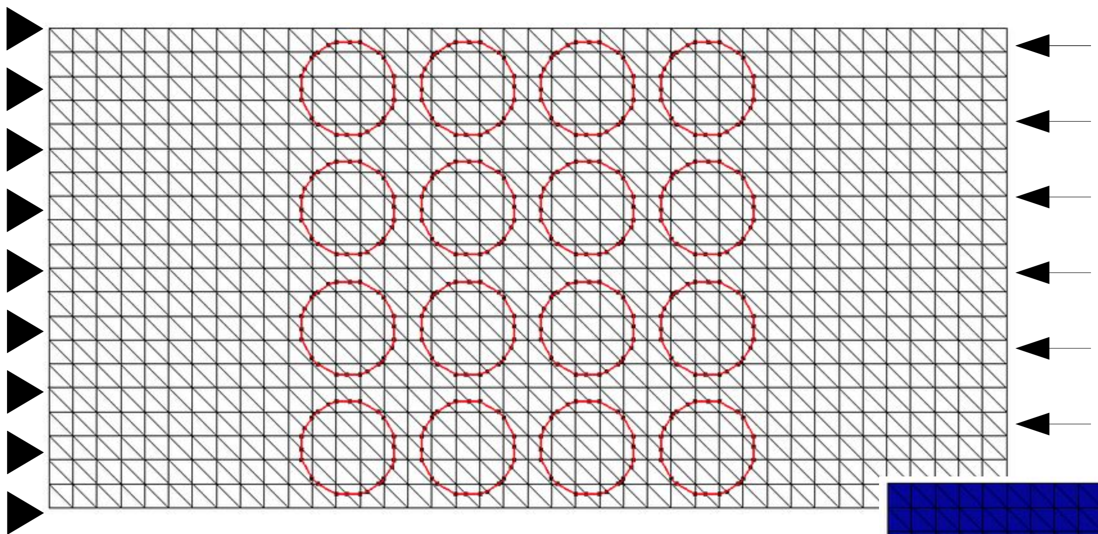
- Direct interfaces with CAD for numerical simulations
  - From an explicit representation to an implicit representation
  - Non conforming boundaries
  - Imposition of boundary conditions
  - Non conforming material interfaces

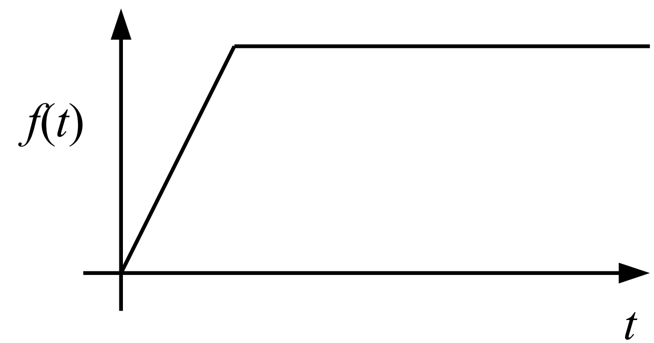
More applications

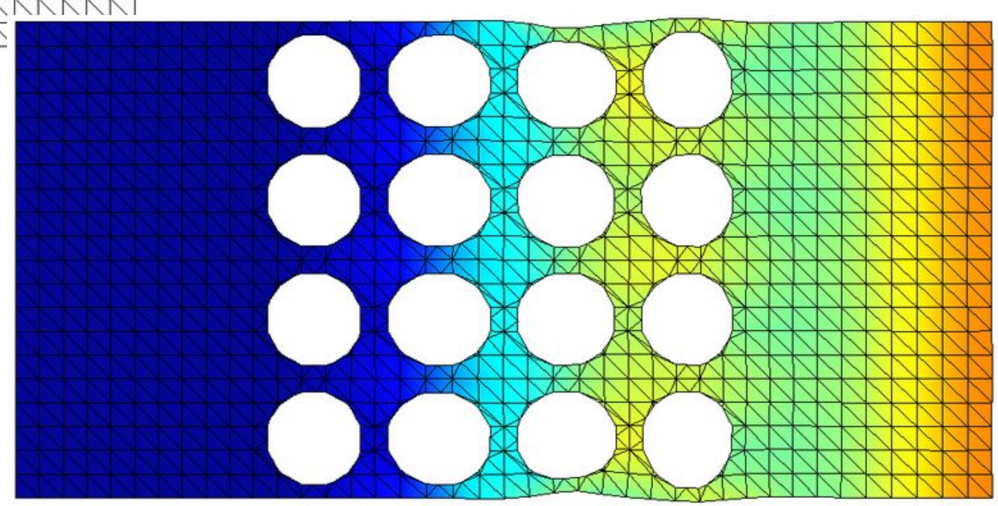
- Applications in explicit dynamics
  - Non conforming geometry → issue with the critical time step
  - Propagation of unstable cracks (change of function space at the crack tip → leads to problems of energy conservation)

More applications

- Explicit dynamics : case without enrichment





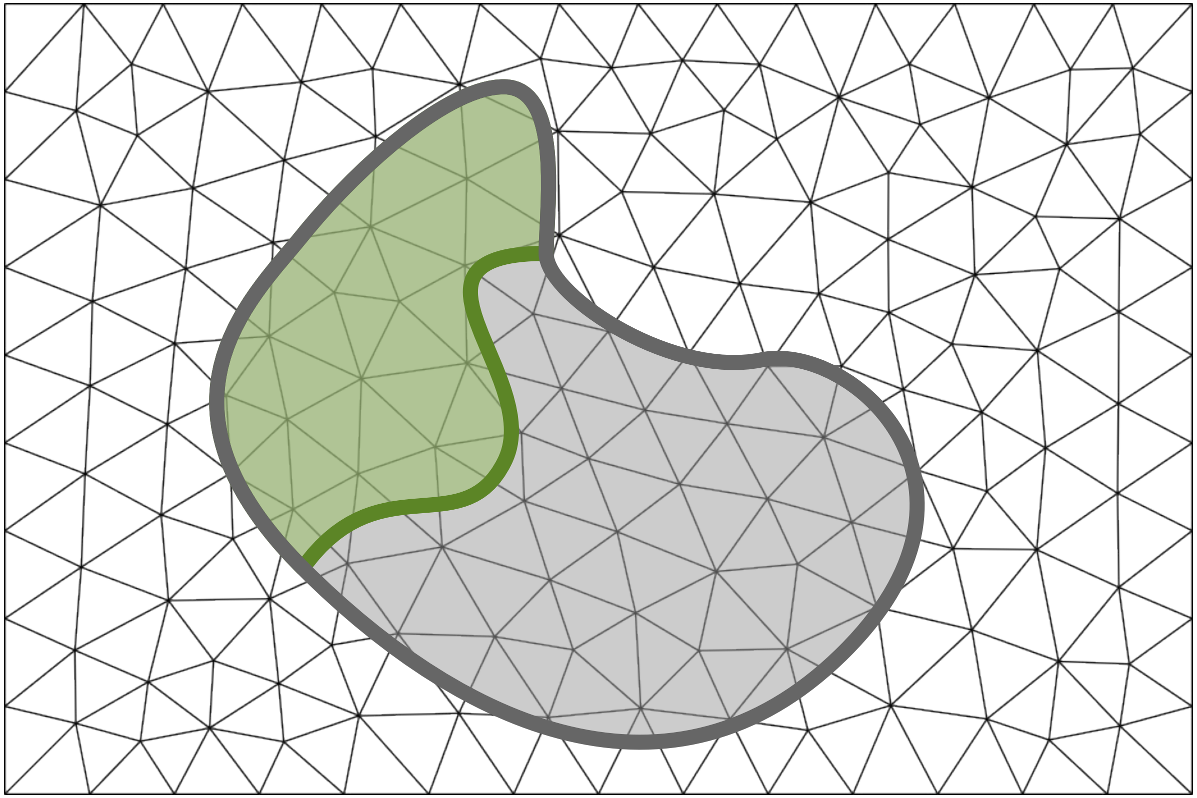


The issue of boundary conditions on implicit volumes

Extended Finite Elements

Goal

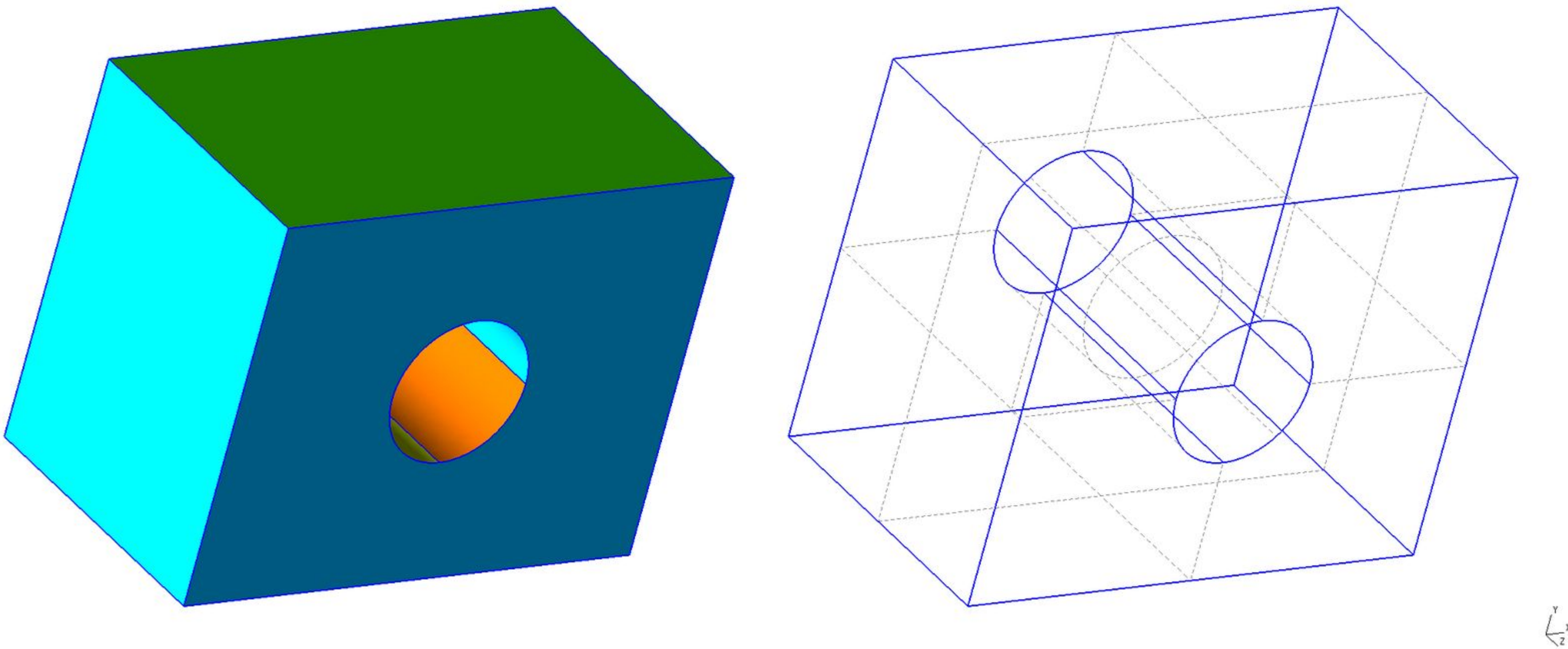
- Free the mesh from geometrical constraints
  - Boundaries of the problem
  - And/or interfaces between different materials



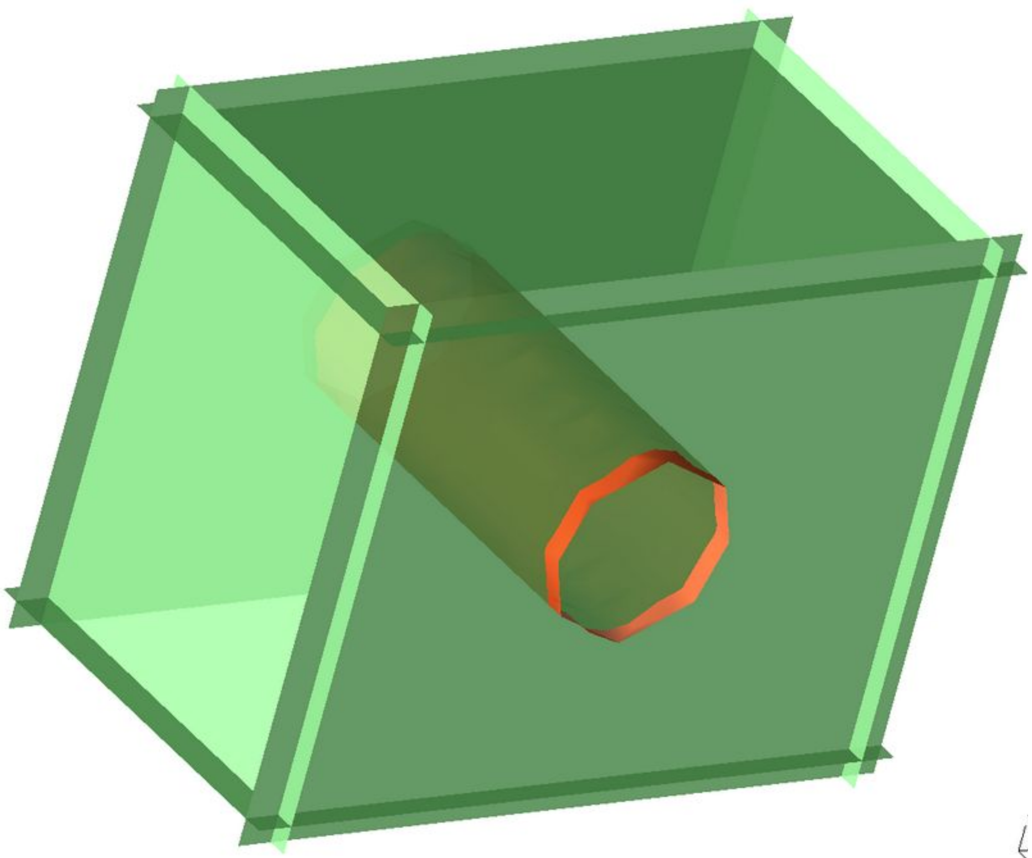
Applications

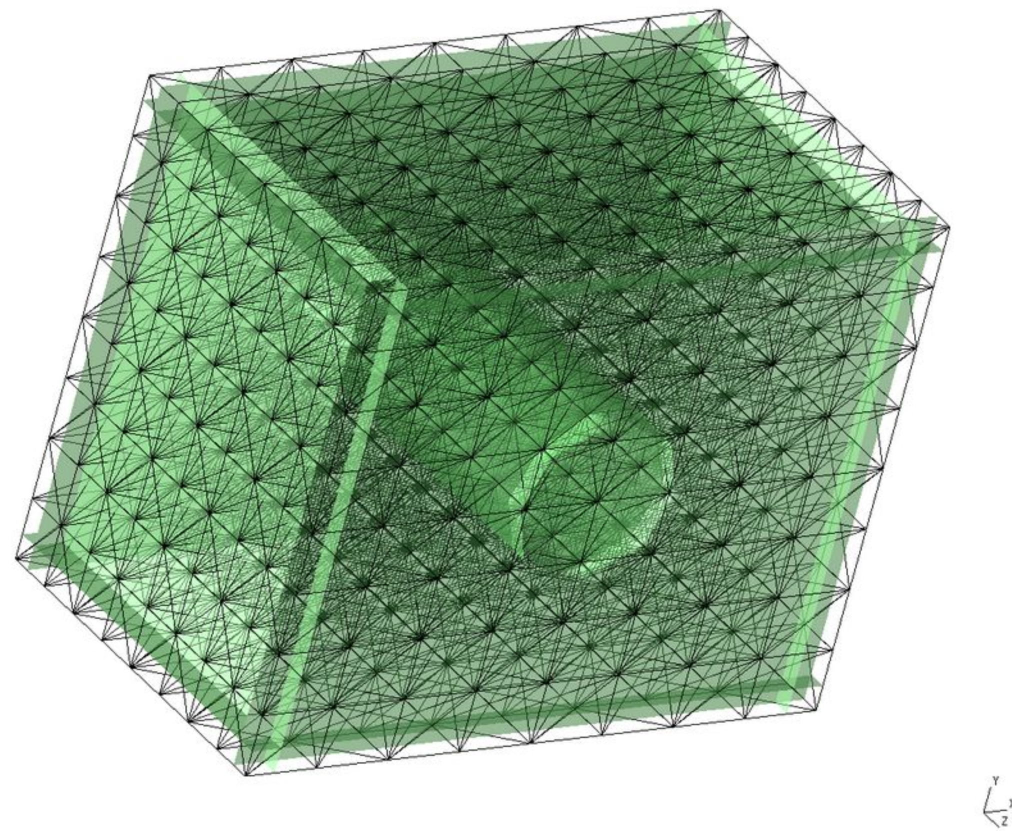
- Direct use of CAD models for the analysis
  - Mesh generation shall be minimalistic
- Use of “dirty” geometrical date not usually adapted to mesh generation
  - Tomography, biomedical applications
- Mobile interfaces
  - Thermoplastic mold filling
  - Topological shape optimization
- Contact problems in mechanical engineering

- From a traditional CAD (B-rep) representation ...



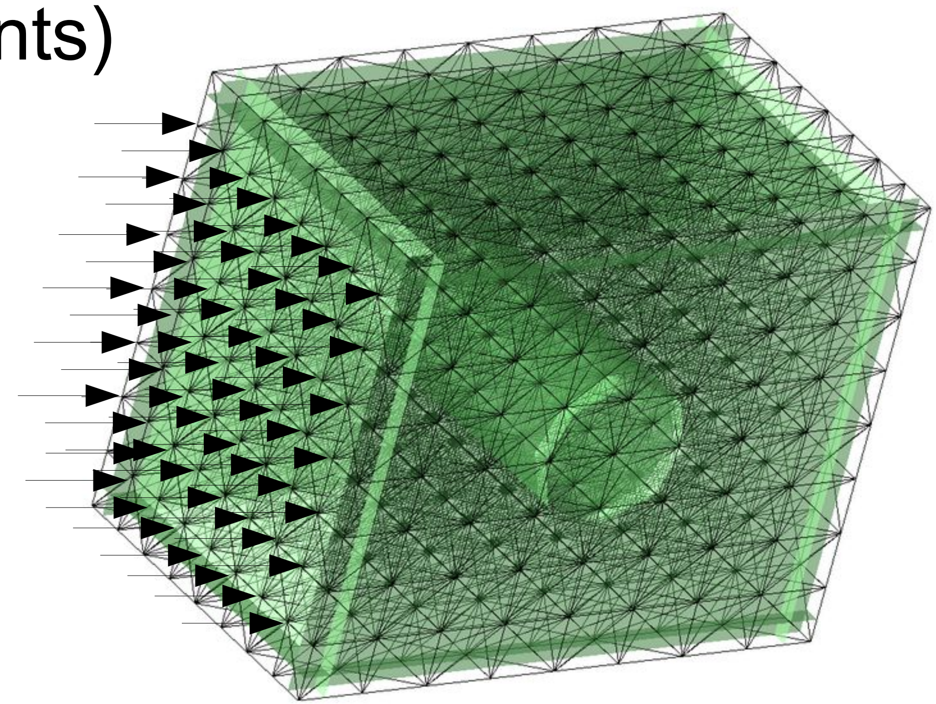
- … To an implicit representation with level-sets





Boundary conditions

- How to apply boundary conditions
  - Neumann/natural boundary conditions (e.g. pressure, forces, gradients)
  - Using integration (it is a linear form)



Find $\bar{u}$ s.t.

$$a(\bar{u},\bar{v}) = b(\bar{v}) \quad \forall \bar{v}$$

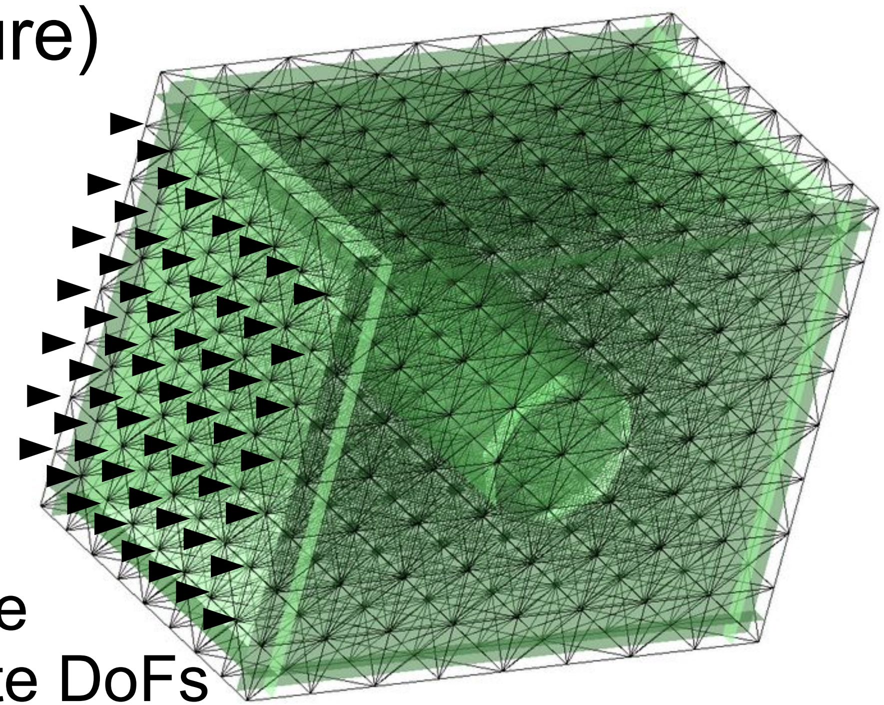
$$a(\bar{u},\bar{v}) = \int_{\Omega} \bar{\bar{\nabla}}^{s}\bar{u} : \bar{\bar{\bar{\bar{D}}}} : \bar{\bar{\nabla}}^{s}\bar{v}\, d\Omega$$

$$b(\bar{v}) = \int_{\Omega} \bar{f}\cdot\bar{v}\, d\Omega + \int_{\Gamma_N} \hat{\bar{f}}\cdot\bar{v}\, d\Gamma_N$$

  - Beware ! The integration is made on a domain Γ_N (or Ω) that cut elements in the mesh

Boundary conditions

- How to apply boundary conditions
  - Dirichlet/essential boundary conditions (e.g. : displacements, temperature)
    - “standard” FEM elimination of DoFs and adding a contribution in the right hand side
    - Here, the domain Γ_D on which to apply this method is non conforming therefore one cannot simply eliminate DoFs - one needs to compute the values to impose at each concerned DoF; so that the “right” Dirichlet BC is obtained on the boundary



- Example 1: a simple Laplacian

$$\text{Find } u \in V_1 = \left\{ v \in H^1(\Omega),\, v_{|\Gamma_D} = u_D \right\} \text{s.t.}$$

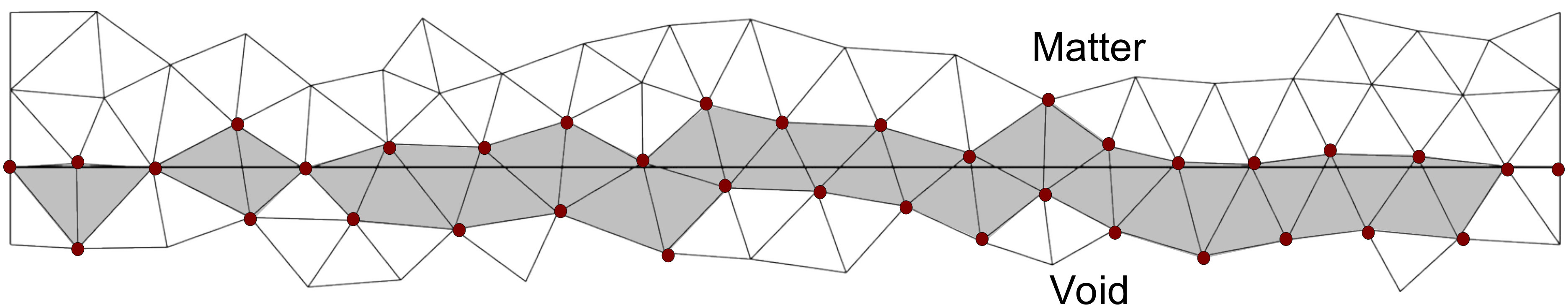
$$a(u, v) = b(v) \quad \forall v \in V_0 = \left\{ v \in H^1(\Omega),\, v_{|\Gamma_D} = 0 \right\}$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega$$

$$b(v) = \int_{\Gamma_N} f \cdot v \, d\Gamma$$

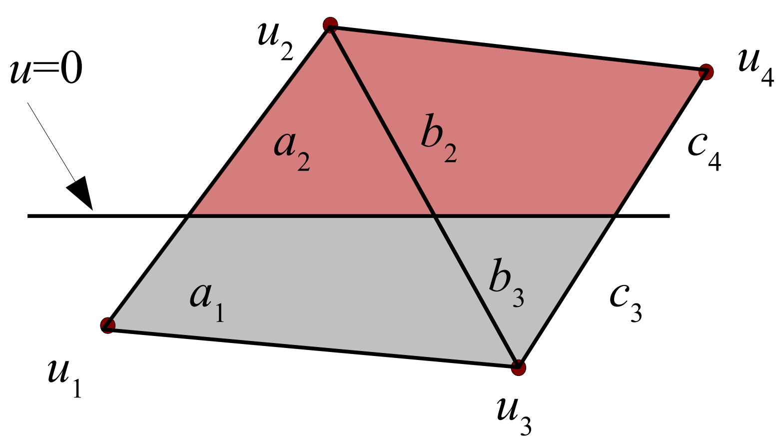
Dirichlet boundary conditions

- Example 1
  - Dofs which are concerned : those where the support cuts the boundary…



Dirichlet boundary conditions

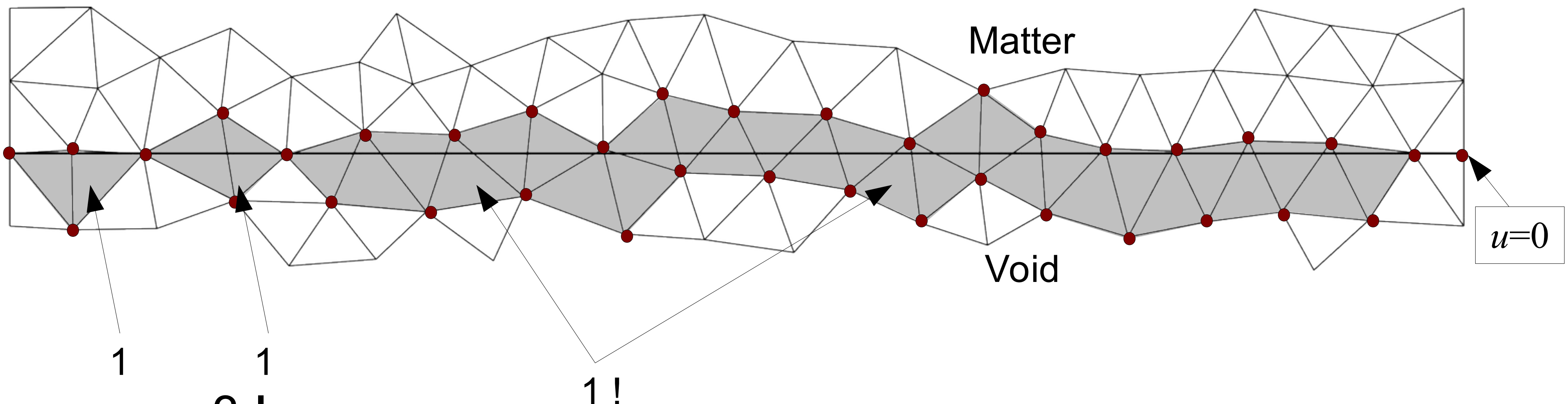
- With only two linear elements ?
  - Without Dirichlet B.C. : 4 DoFs , u has some freedom in the red part
  - If one imposes *exactly* $u=0$ on the boundary …



- $$\frac{u_1}{a_1}=\frac{u_2}{a_2}; \quad \frac{u_2}{b_2}=\frac{u_3}{b_3}; \quad \frac{u_3}{c_3}=\frac{u_4}{c_4}$$
- How many DoFs left for the red part of the domain ?

Dirichlet boundary conditions

- Concrete example
  - Number of available DoFs after imposing *exactly* the Dirichlet B.C. :



  - 3 !
  - The function space is very poor in the elements crossed by the interface, therefore the F.E. solution will be far from accurate.

Dirichlet boundary conditions

- One cannot impose exactly a Dirichlet B.C. by elimination as long as it is crossing through finite elements !
- For this, an interpolation is preferred and the B.C. must be along element edges.
- This is the reason why Lagrange F.E. are so widely used.
  - (One) solution : the use of Lagrange multipliers, see an article of Babuska (1973) - in the bibliography)

Lagrange multipliers

On wants to minimize $\pi(u,v)=u^2+v^2$

$$\delta\pi(u,v)=2u\,\delta u+2v\,\delta v=0 \qquad \forall\,\delta u,\delta v$$

$$u=v=0 \qquad \pi(0{,}0)=0$$

If one sets an additional condition :

$$g(u,v)=u-v+2=0$$

Method 1 : elimination of v :

$$\pi'(u)=2u^2+4u+4\equiv\pi(u,v)$$

$$\delta\pi'(u)=4(u+1)\,\delta u=0 \qquad \forall\,\delta u$$

$$u=-1 \qquad \pi'(-1)=2 \rightarrow v=1$$

This is the method used just before ...

Lagrange multipliers

Method 2 : Introduction of an additional variable

$$\tilde{\pi}(u,v,\lambda)=\pi(u,v)+\lambda\, g(u,v)=u^2+v^2+\lambda(u-v+2)$$

$$\begin{aligned}\delta\tilde{\pi}(u,v,\lambda)&=0\\ &=(2u+\lambda)\delta u+(2v-\lambda)\delta v+(u-v+2)\delta\lambda \qquad \forall\,\delta u,\delta v,\delta\lambda\end{aligned}$$

$$\begin{cases}2u+\lambda=0\\ 2v-\lambda=0\\ u-v+2=0\end{cases}\Leftrightarrow\begin{cases}u=-1\\ v=1\\ \lambda=2\end{cases}$$

- In finite elements, this gives us :

$$a(u,\delta u)=l(\delta u)$$

$$-\Delta u = f \text{ in } \Omega$$
$$u = u_D \quad \text{on } \Gamma_D$$

of weak form:
find u s.t.

$$\int_{\Omega} \nabla u \cdot \nabla \delta u \, d\Omega = \int_{\Omega} f \, \delta u \, d\Omega \quad \forall \delta u$$

Equivalent to minimize $F(u)=\frac{1}{2}\int_{\Omega} \nabla u \cdot \nabla u \, d\Omega - \int_{\Omega} f u \, d\Omega$
if the conditions of
Lax-Milgram's theorem are satisfied. $=\frac{1}{2}a(u,u)-l(u)$

, for all u satisfying the B.C. on Γ_D. By using Lagrange multipliers for the BC's, one gets a new functional to minimize:

$$\tilde{F}(u,\lambda)=\frac{1}{2}\int_{\Omega} \nabla u \cdot \nabla u \, d\Omega - \int_{\Gamma_D} \lambda (u-u_D) \, d\Gamma_D - \int_{\Omega} f u \, d\Omega$$

Lagrange multipliers

- Associated weak form :

$$\widetilde{F}(u,\lambda)=\frac{1}{2}\int_{\Omega}\nabla u\cdot\nabla u\,d\Omega-\int_{\Gamma_D}\lambda(u-u_D)\,d\Gamma_D-\int_{\Omega}fu\,d\Omega$$

$$=\frac{1}{2}A(U,U)-L(U) \qquad U=\begin{pmatrix}u\\ \lambda\end{pmatrix}$$

$$A(U,U)=(u,\lambda)\cdot\begin{pmatrix}a & b\\ b & 0\end{pmatrix}\cdot\begin{pmatrix}u\\ \lambda\end{pmatrix}=a(u,u)+b(u,\lambda)+b(\lambda,u)$$

$$L(U)=l(u)+c(\lambda)$$

$$a(u,v)=\int_{\Omega}\nabla u\cdot\nabla v\,d\Omega$$

$$A(U,\delta U)=L(\delta U)$$

$$b(u,\lambda)=b(\lambda,u)=-\int_{\Gamma_D}u\cdot\lambda\,\Gamma_D$$

$$\downarrow$$

$$l(u)=\int_{\Omega}fu\,d\Omega$$

$$a(u,\delta u)+b(\lambda,\delta u)=l(\delta u)$$
$$b(\delta\lambda,u)=c(\delta\lambda)$$

$$c(\lambda)=-\int_{\Gamma_D}u_D\,d\Gamma_D$$

Extended Finite Elements

Dirichlet boundary conditions

To simplify notations, lets assign $v=\delta u\,,\mu=\delta\lambda$

$$\text{Find}\ \begin{matrix} u\in V=\left\{v\in H^1(\Omega)\right\} \\ \lambda\in L=\left\{\mu\in H^{1/2}(\Gamma_D)'\right\} \end{matrix}\ \text{s. t.}$$

$$\int_{\Omega}\nabla u\cdot\nabla v\,d\Omega-\int_{\Gamma_D}\lambda\cdot v\,d\Gamma=\int_{\Gamma_N}f\cdot v\,d\Gamma \qquad \forall v\in V$$

$$-\int_{\Gamma_D}\mu\cdot u\,d\Gamma=\boxed{-\int_{\Gamma_D}\mu\cdot u_D\,d\Gamma} \qquad \forall\mu\in L$$

The Dirichlet B.C. has been "dualized".
This is now a Neumann B.C. on the lagrange multipliers

Dirichlet boundary conditions

- The Lagrange multipliers have a physical meaning
  - In mechanics, it is the force to impose so that the condition on the primal variable is ensured (here, displacements).
  - In our case, it is the gradient of the solution (flux) to impose so that $u=u_D$ on Γ_D.
- We have now a saddle point problem (min-max) – the matrix of the linear system is not definite positive anymore (but still has an inverse and is symmetric)
- Not all solvers are able to handle that – mostly direct solvers and very few iterative solvers.

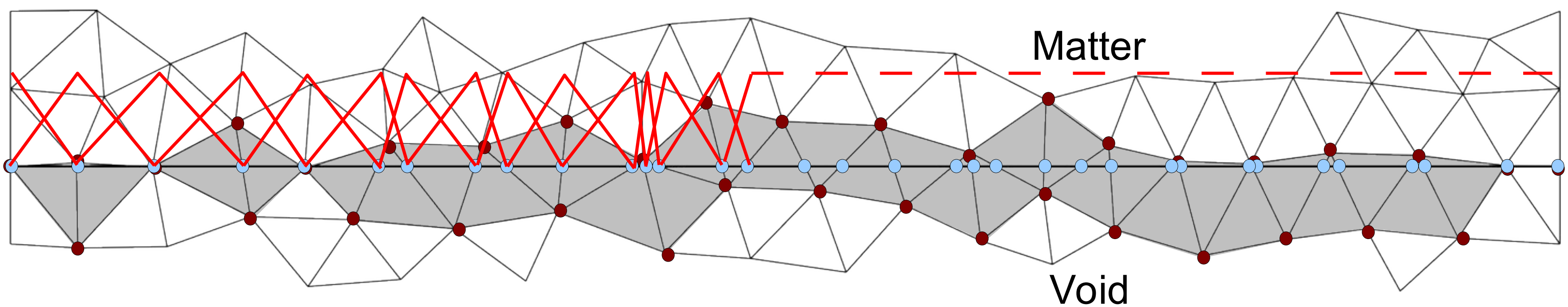
Dirichlet boundary conditions

- How to build adequate discrete function spaces
  - Find $\begin{matrix} u_h \in V_h \subset V = \left\{ v \in H^1(\Omega) \right\} \\ \lambda_h \in L_h \subset L = \left\{ \mu \in H^{1/2}(\Gamma_D)' \right\} \end{matrix}$ s. t. ...
  - One do not change the primal functional space (for u). It is the usual finite element space using nodal hat functions
  - One need to build a function space for λ.
    - Lets try to use an identical function space L_h for λ (or the restriction to the boundary of such a space… (the trace)

Extended Finite Elements

Dirichlet boundary conditions

- Lets try to use an identical function space L_h for λ (or the restriction to the boundary of tsuch a space… (the trace)



- Lets perform a computation. The linear system has the following shape :

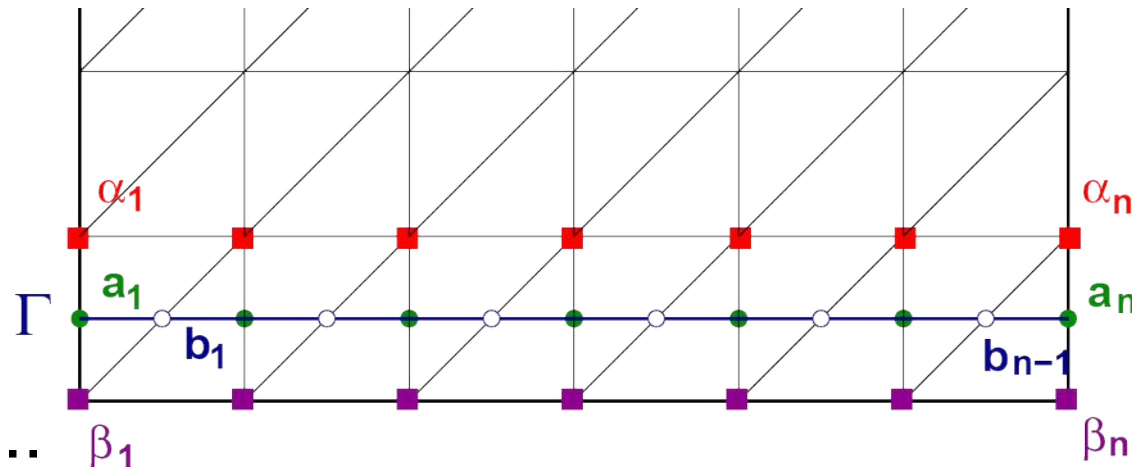
$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \qquad \forall v \in V_h$$

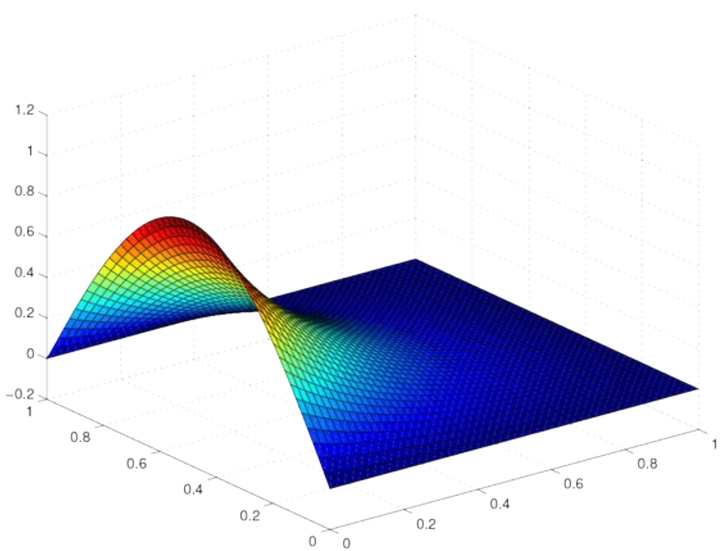
$$-\int_{\Gamma_D} \mu \cdot u \, d\Gamma = -\int_{\Gamma_D} \mu \cdot u_D \, d\Gamma \qquad \forall \mu \in L_h$$

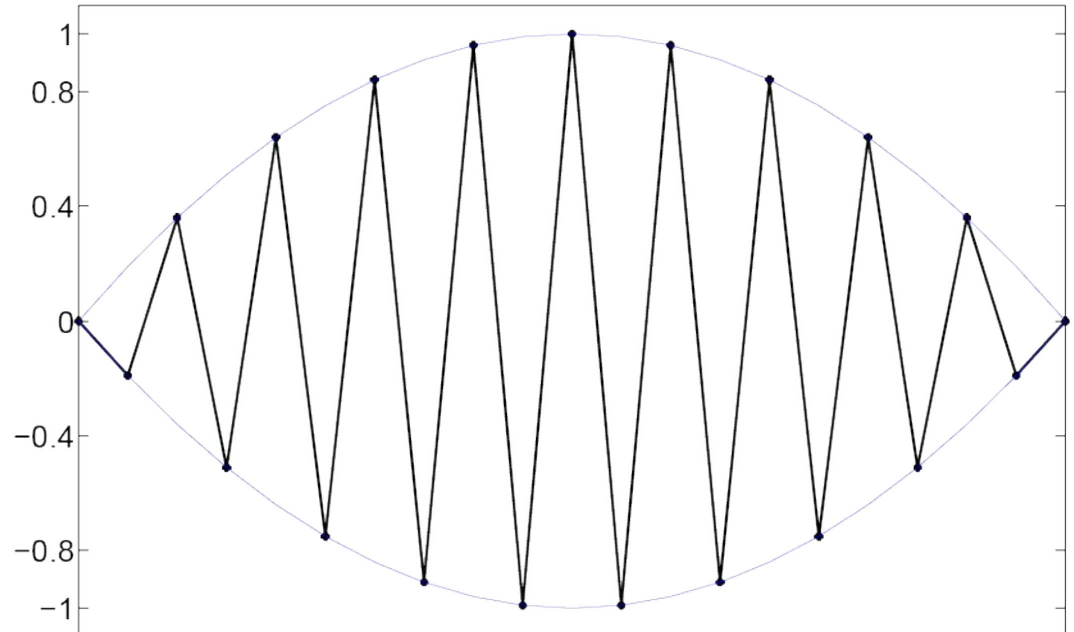
$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}$$

Dirichlet boundary conditions

- The we solve it …
  - Lagrange multipliers are oscillating.
  - The more h (element size) shrinks, the more it oscillates…







Dirichlet boundary conditions

- What happens ?
  - The discrete spaces for u et λ are incompatible.
  - Those do not satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, or inf-sup condition :

$$\inf_{\mu \in L_h} \sup_{u \in V_h} \frac{\int_\Gamma \lambda_h u_h \, d\Gamma}{h^{1/2} \|\lambda\|_{0,\Gamma_D} \|u\|_{1,\Omega}} \geq \alpha > 0$$

  - This condition is often difficult to check analytically.

O. Ladyzhensakya, Global solvability of a boundary value problem for the Navier–Stokes equations in the case of two spatial variables. *Proc. Ac. Sc. USSR* 123 (3) (1958) 427–429.
I. Babuska, Error bounds in the finite element method, *Numer. Math.*, 16 (1971), pp. 322-33.
F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, RAIRO, Anal. Num., 8, R2 (1974), pp. 129-151

Dirichlet boundary conditions

- Numerical validation of the LBB condition.
  - There exists a “simple” numerical test; see Chapelle, Bathe, 1993 and KJ Bathe 2001 (in the bibliography)
    - One considers a more general problem with an added “stiffness” on the Dirichlet boundary condition (becomes a Robin B.C.) – if $k \rightarrow \infty$, back to a “hard” Dirichlet B.C.

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \qquad \forall v \in V_h$$

$$-\int_{\Gamma_D} \mu \cdot u \, d\Gamma - \boxed{\int_{\Gamma_D} \frac{1}{k} \lambda \mu \, d\Gamma} = -\int_{\Gamma_D} \mu \cdot u_D \, d\Gamma \qquad \forall \mu \in L_h$$

$$\begin{pmatrix} A_h & B_h^T \\ B_h & \boxed{-\frac{1}{k} M_h} \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}$$

Dirichlet boundary conditions

- Chapelle – Bathe numerical test

$$\begin{pmatrix} A_h & B_h^T \\ B_h & -\frac{1}{k} M_h \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}$$

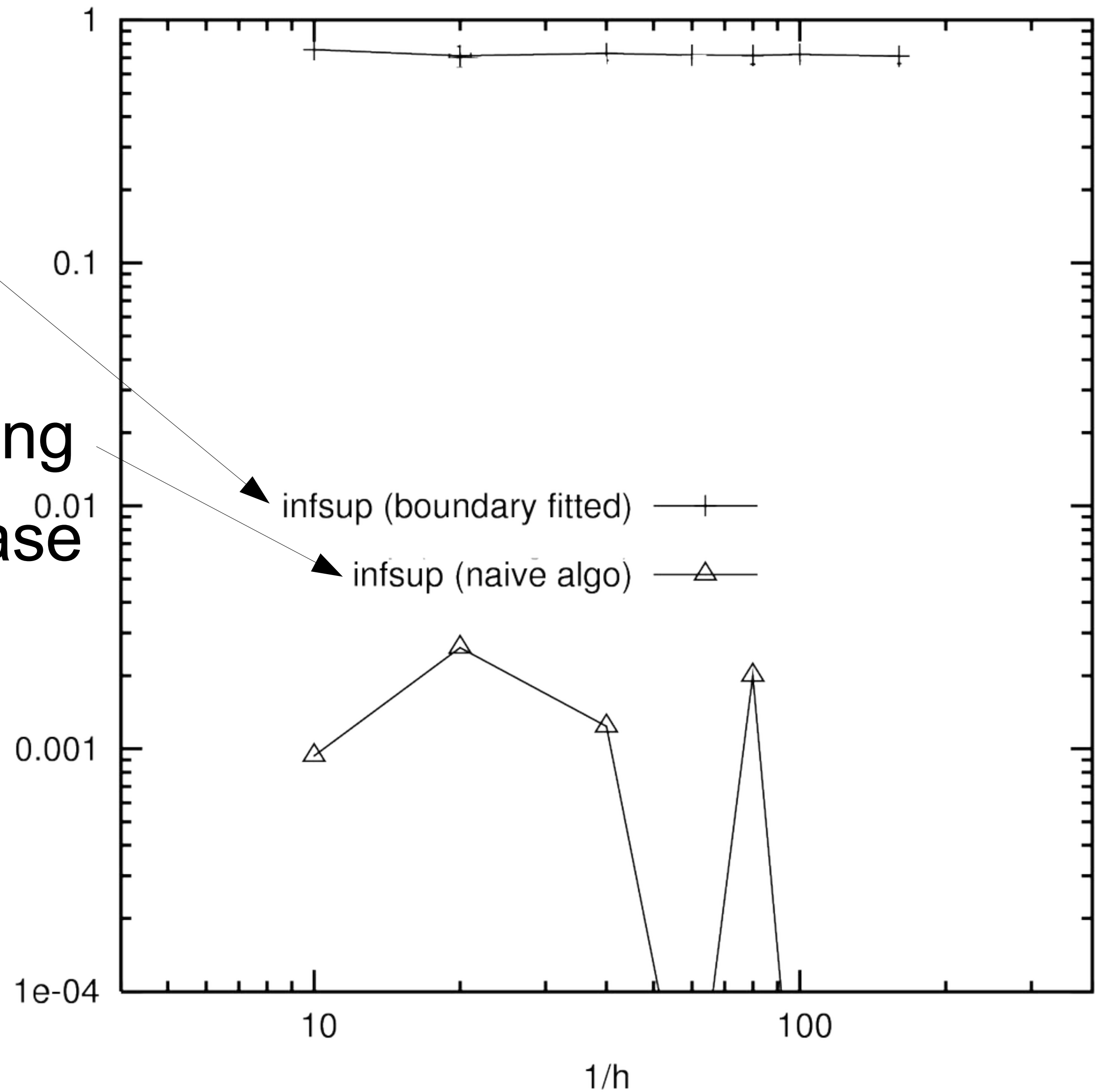
  - It amounts to check the first non vanishing eigenvalue (β_0) of the following eigenproblem :
    $\frac{1}{h}\left(B_h A_h^{-1} B_h^{\mathrm{T}}\right) W_h = \beta M_h W_h$ ou $\frac{1}{h}\left(B_h^{\mathrm{T}} M_h^{-1} B_h\right) W'_h = \beta' A_h W'_h$
  - A_h must have an inverse
  - Does not depend on k !
  - One checks that β_0 does not vanish for a sequence of meshes with an increasing density
  - Here, $\alpha = \sqrt{\beta_0}$ (and for α : see slides before)

Dirichlet boundary conditions

- Results
  - Two cases :

    - aligned with the mesh

    - non conforming
  - The second case does not work at all.



Dirichlet boundary conditions

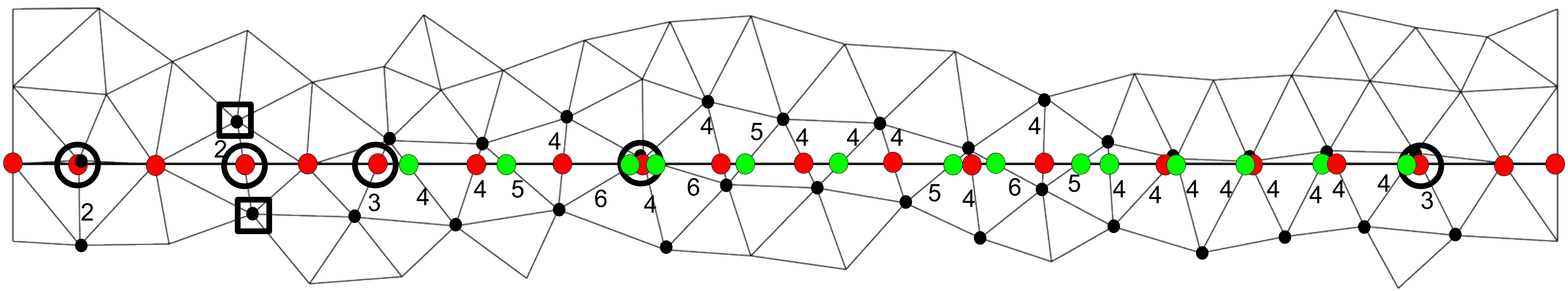
- What we have are incompatibles functional spaces...
  - The space for the Lagrange multipliers is way too “rich” with respect to the one for the primal variable.
  - It amounts to impose exactly the Dirichlet B.C., which has been already shown to be a bad idea.

  $\rightarrow$ We have to “decimate” L_h

Extended Finite Elements

Dirichlet boundary conditions

- From the mesh of the interface, take each node and put it in a set N
- If a node of N is also part of the mesh, mark it as Vital (set V) , and delete it from N
- Take each edge incident to N and count each intersecting edge going from end nodes with the interface
- Sort N. The sorting key is the number defined above (smallest first)



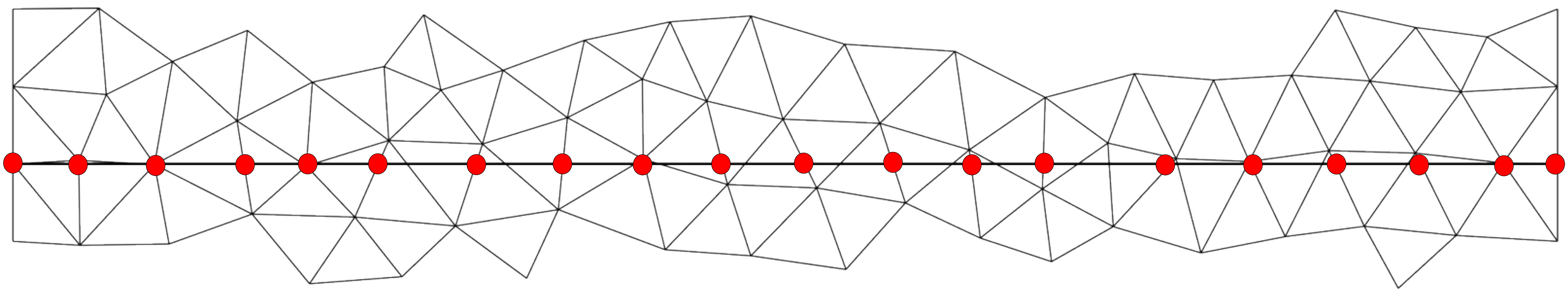
- Loop over the sorted set N, take ni
  - Take the end nodes of ni, and from those, the connected nodes in N (may be many)
  - If ni is not yet NV (non vital), mark it as Vital (V) and all the other connected nodes as (NV)
- EndLoop

Extended Finite Elements

Dirichlet boundary conditions

What remains,

- An approximately uniform distribution of nodes

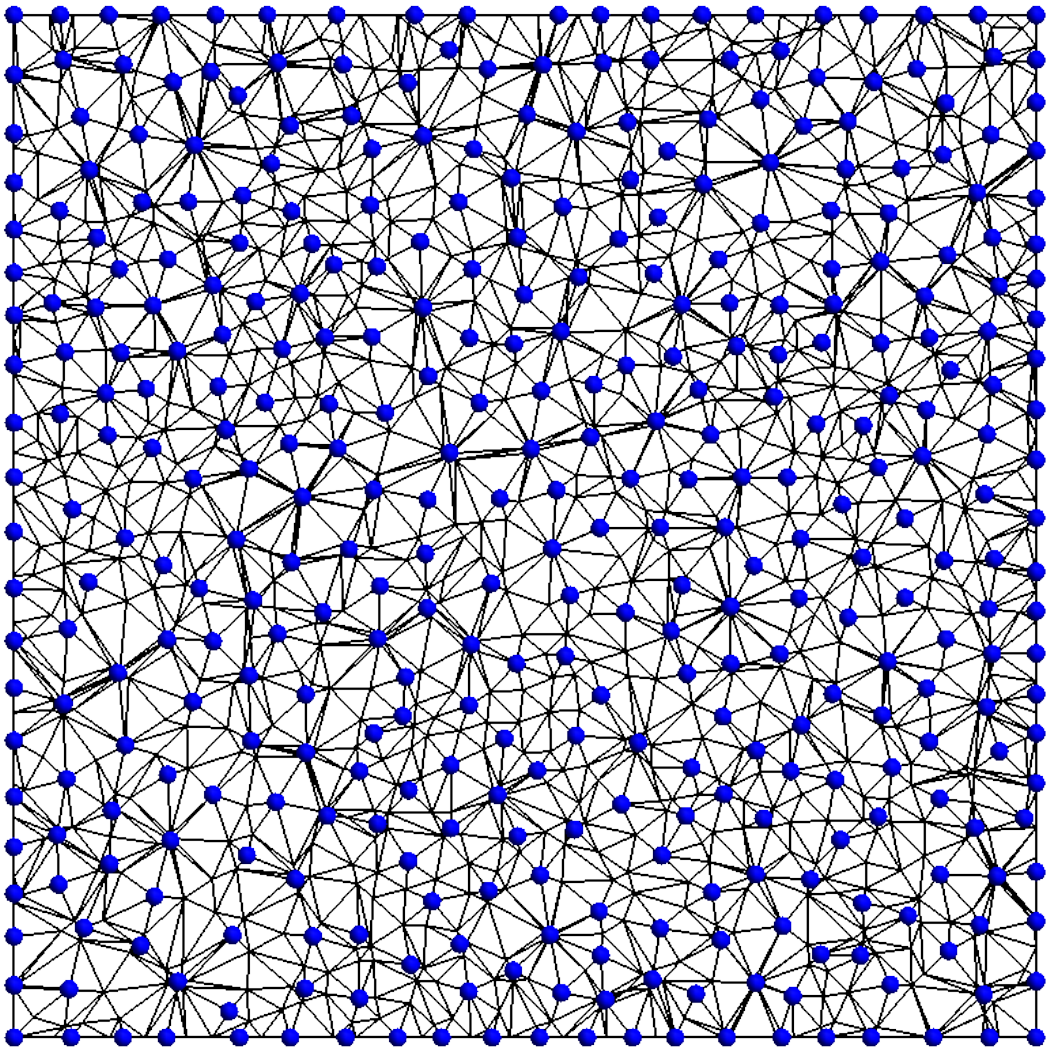


- The density is same as the initial mesh (2D here, 3D in general
- Works in 3D !

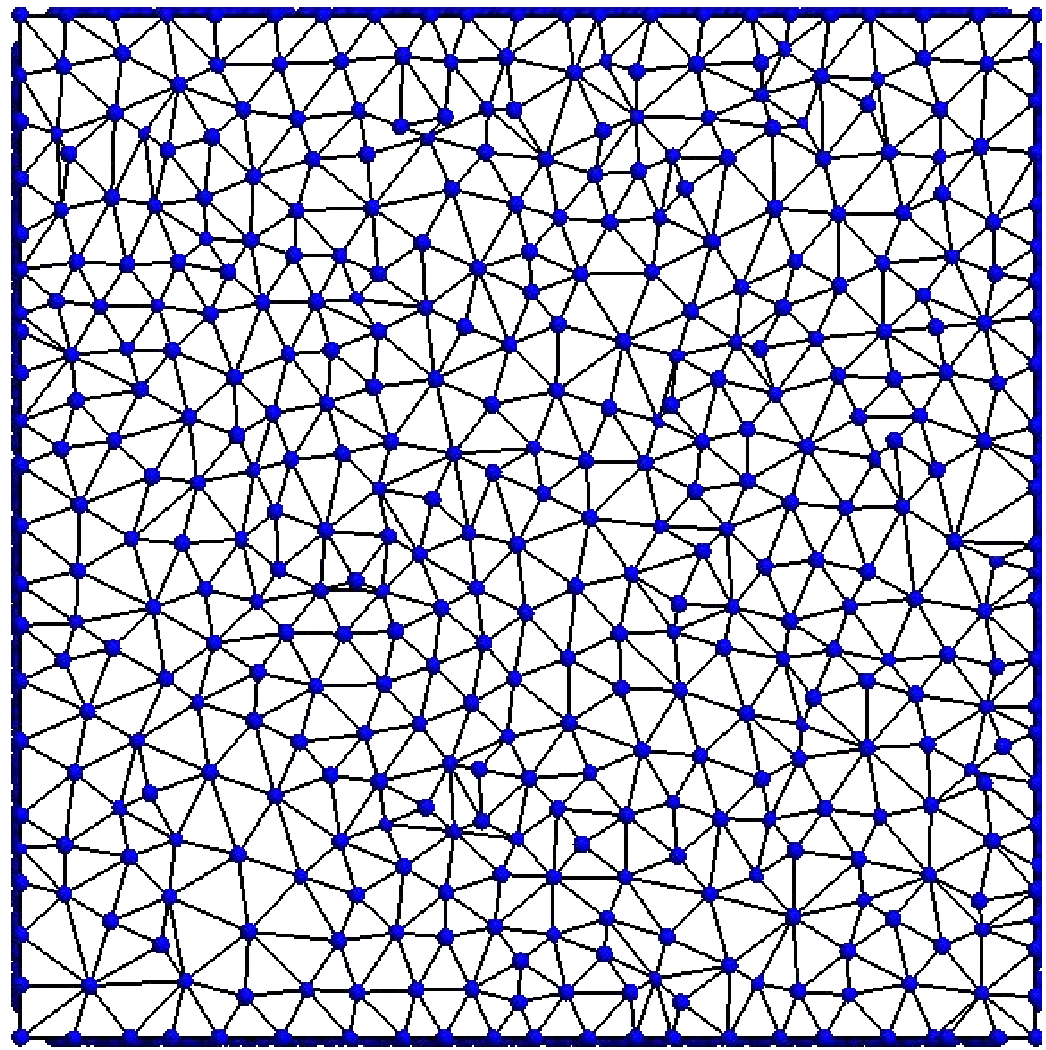
Extended Finite Elements

Dirichlet boundary conditions

Result of the decimation



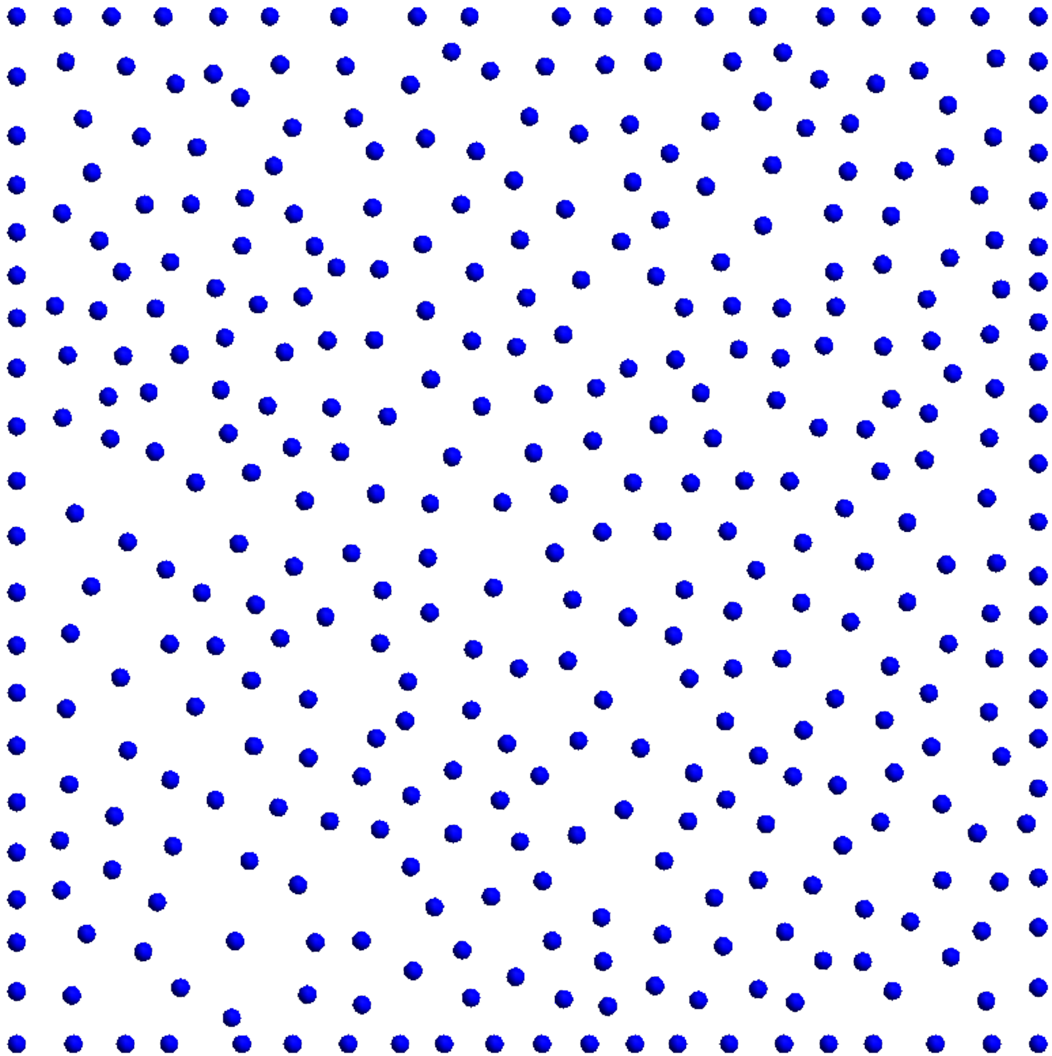
Projection of 3D nodes



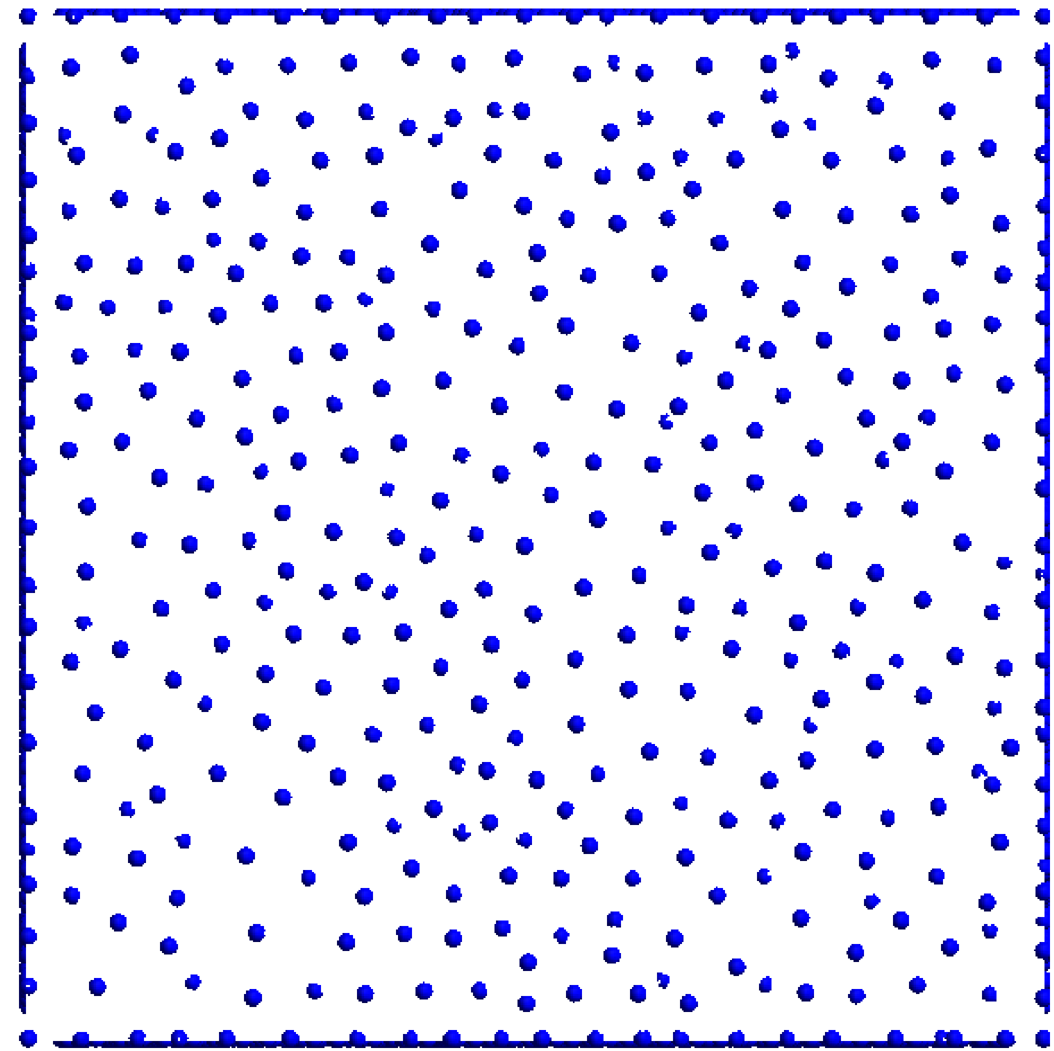
Extended Finite Elements

Dirichlet boundary conditions

Result of the decimation

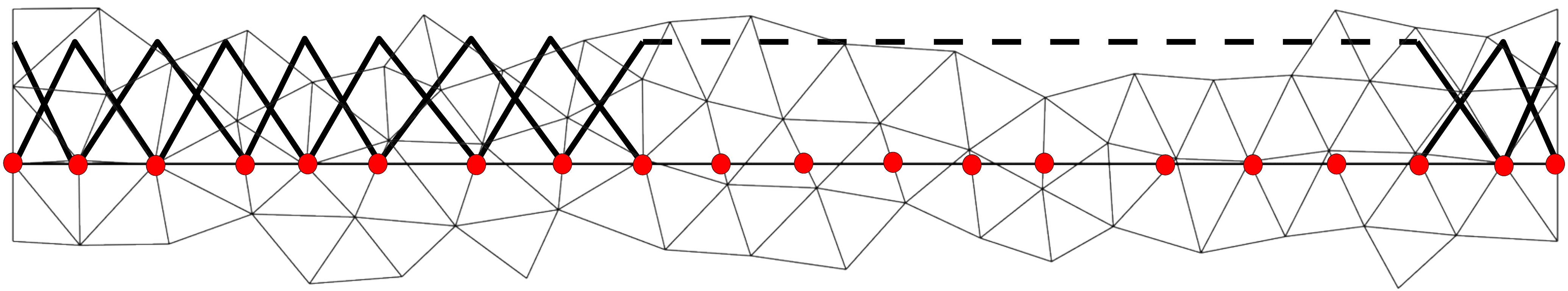


Projection of 3D nodes



Dirichlet boundary conditions

- How to build shape functions from this ?
  - Directly on the interface ?



  - Works...

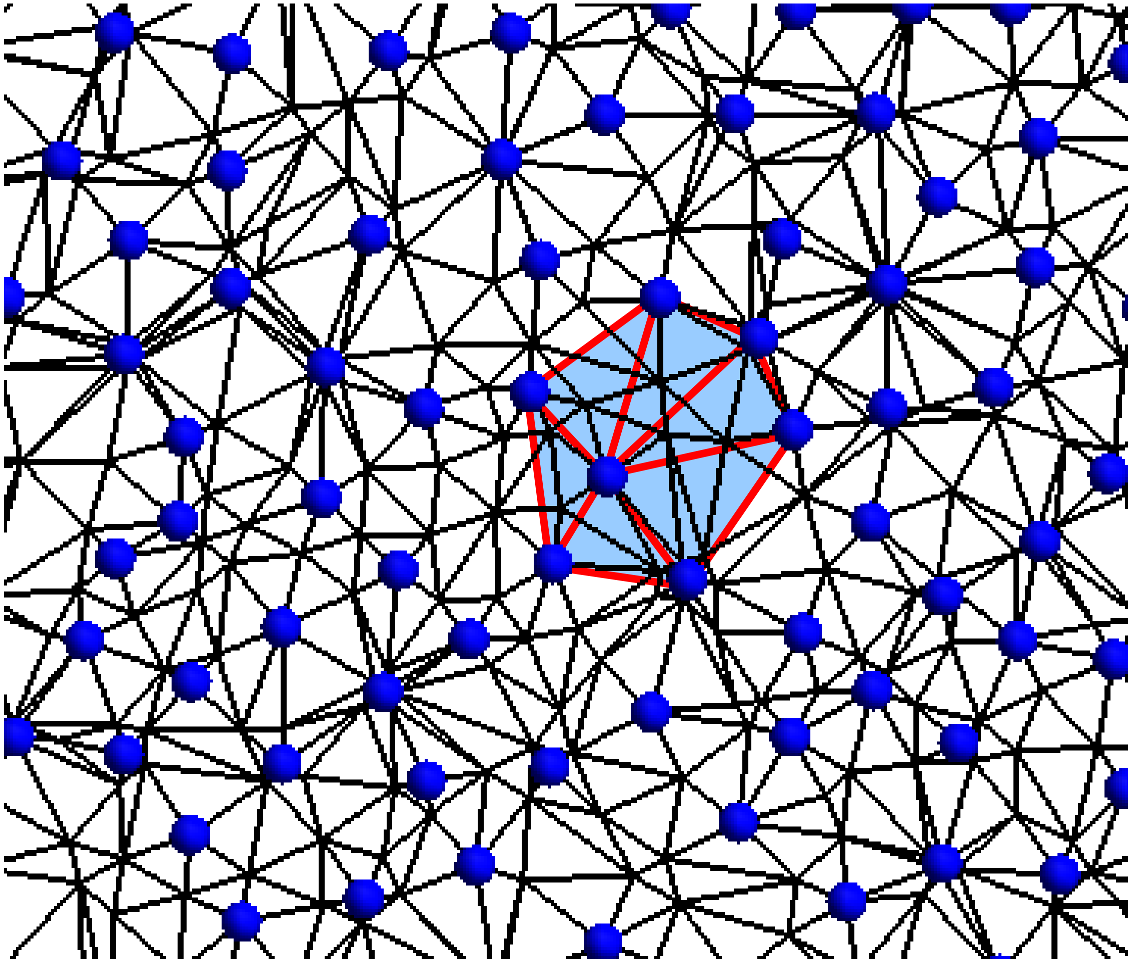
    … only in 2D !!!

Dirichlet boundary conditions

- In 3D : one would have to build a triangulation of the set of nodes V

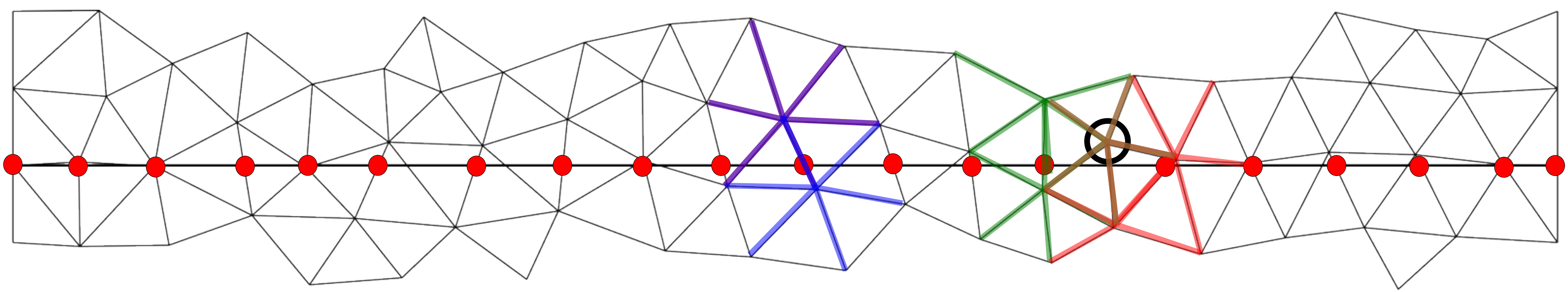
What about :

- Curvy interfaces
- Discrepancy (non conformity) btw. triangulations
  - Integration problems
- So we must find a better way in 3D…



Dirichlet boundary conditions

- Another solution
  - Lets take the trace of volume shape functions – but there are too many !
    - One will combine SFs. (linear combinations) for each V-node

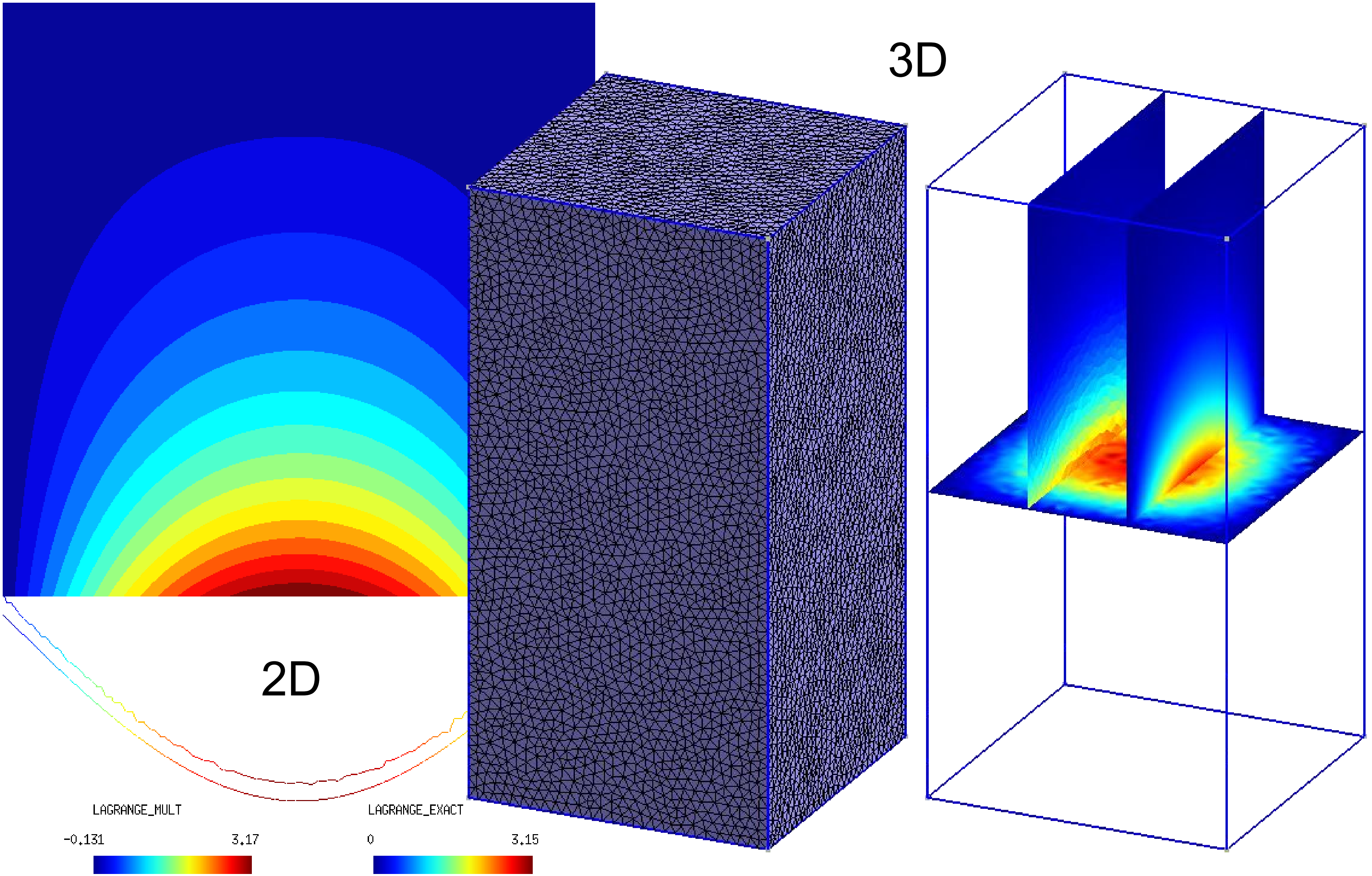


    - At some places, a volume SF may be linked to more than one V-node.
    - There is room for freedom : 100% with the green, or 100% with the red or whatever combination such that the sum is 100% (to keep “partition of unity”)

Dirichlet boundary conditions

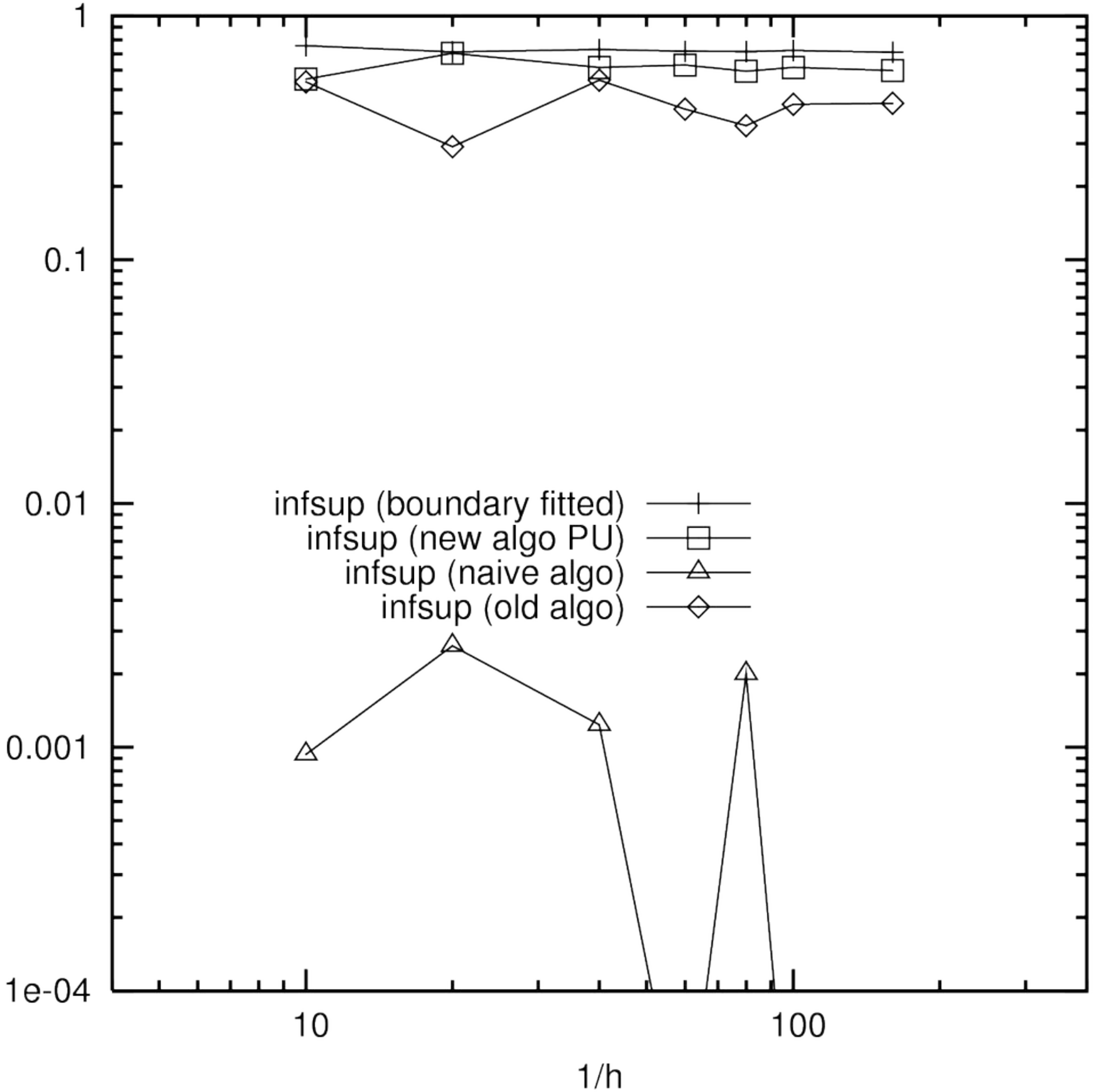
- Advantages of using trace shape function for Lagrange multipliers
  - Easy integration
  - Compact shape functions
  - Partition of unity on the interface
  - Same algorithm in 3D and 2D
  - Good numerical results ? See what’s follow !

Dirichlet boundary conditions



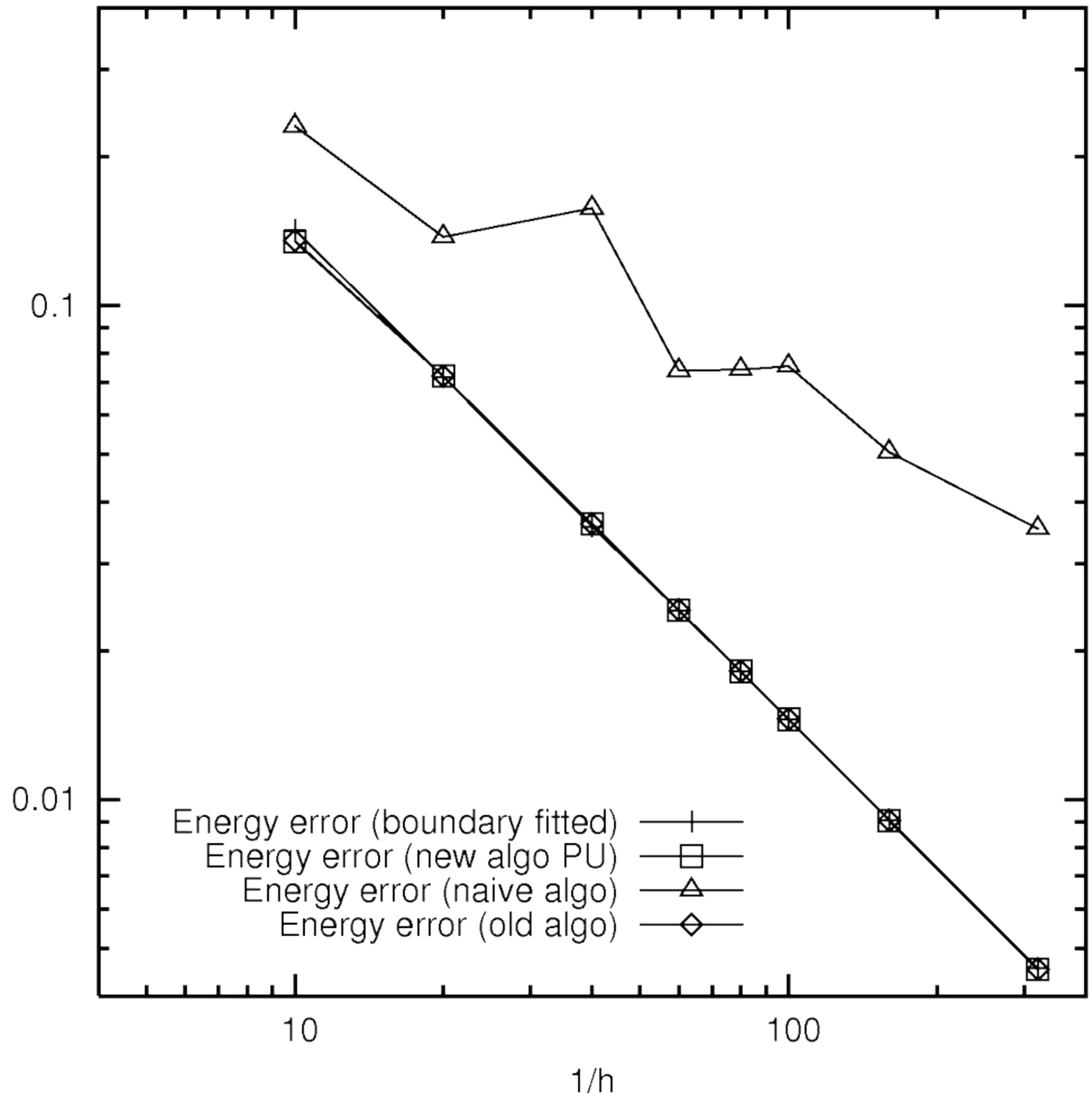
Extended Finite Elements

Dirichlet boundary conditions



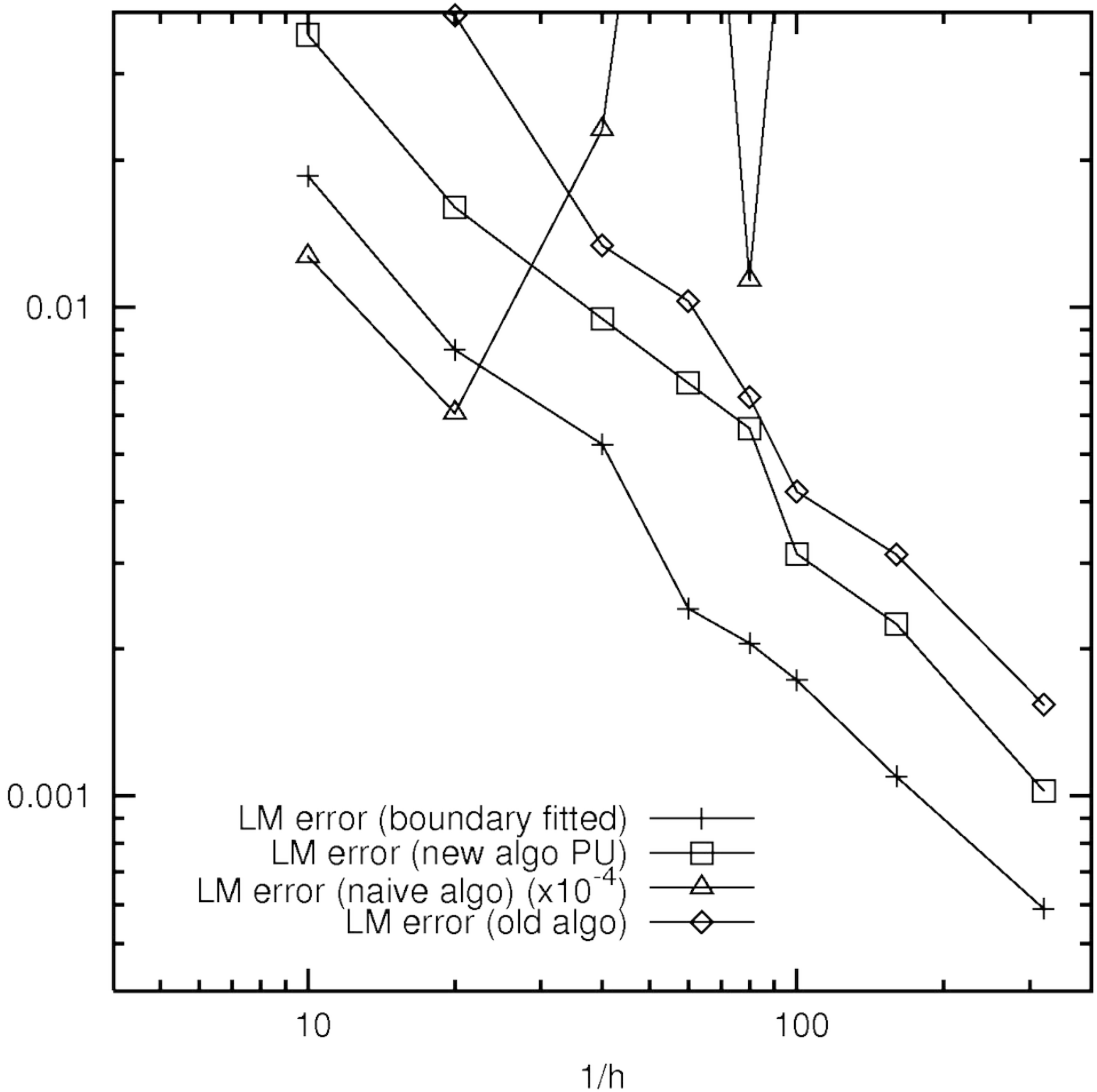
Extended Finite Elements

Dirichlet boundary conditions



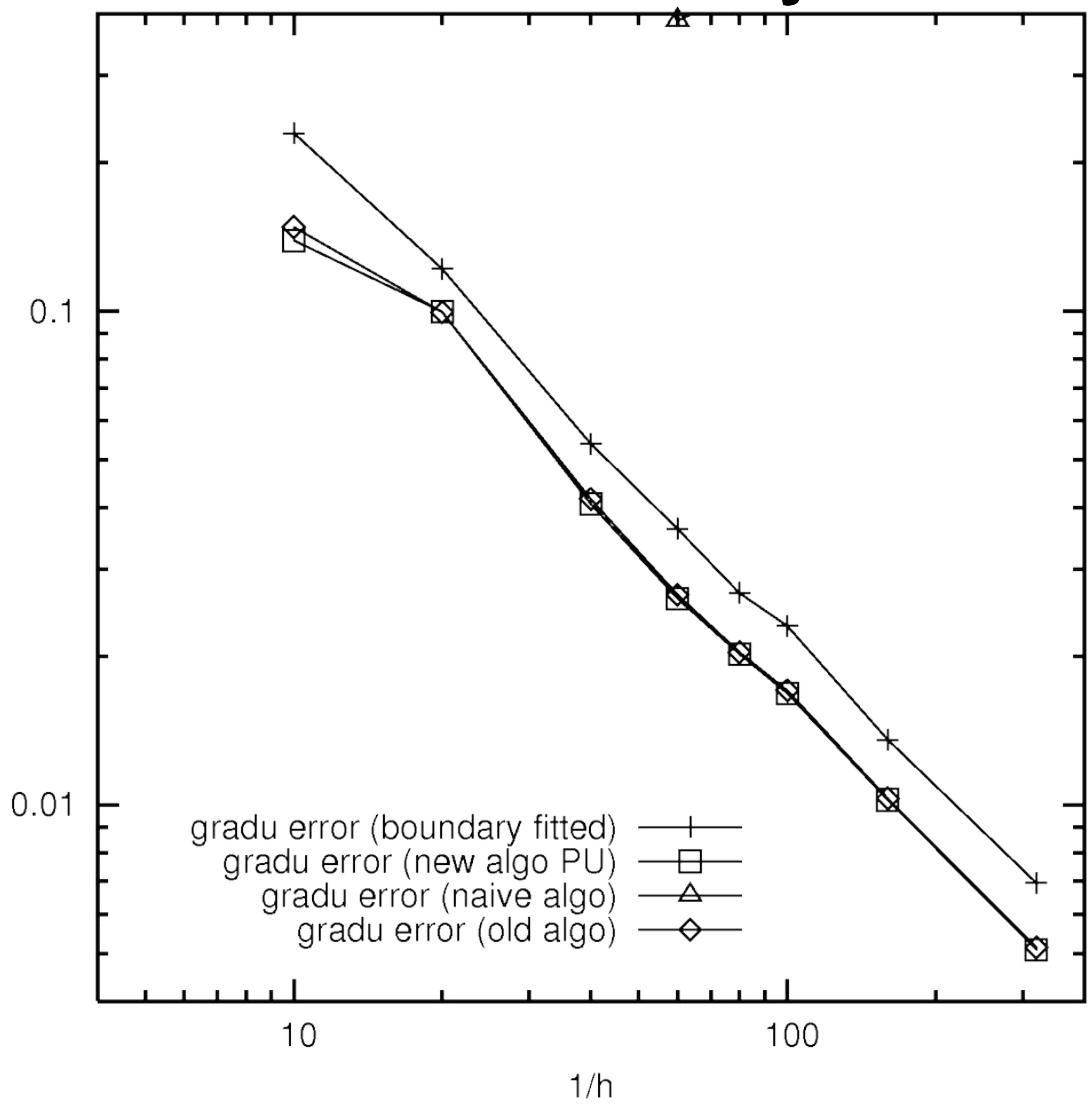
Extended Finite Elements

Dirichlet boundary conditions



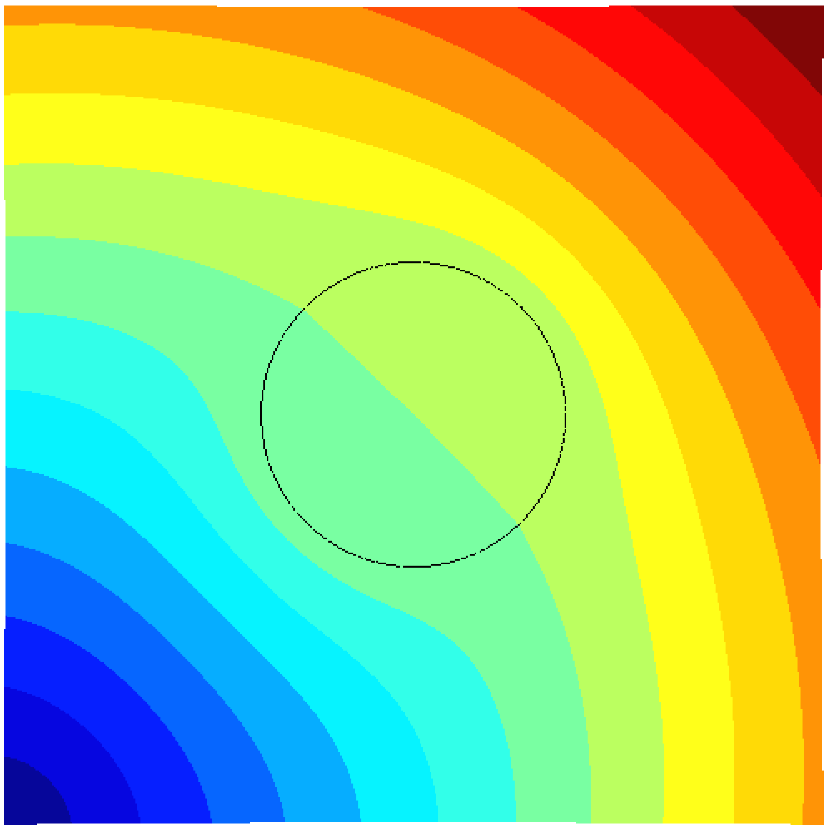
Extended Finite Elements

Dirichlet boundary conditions



Extended Finite Elements

Dirichlet boundary conditions

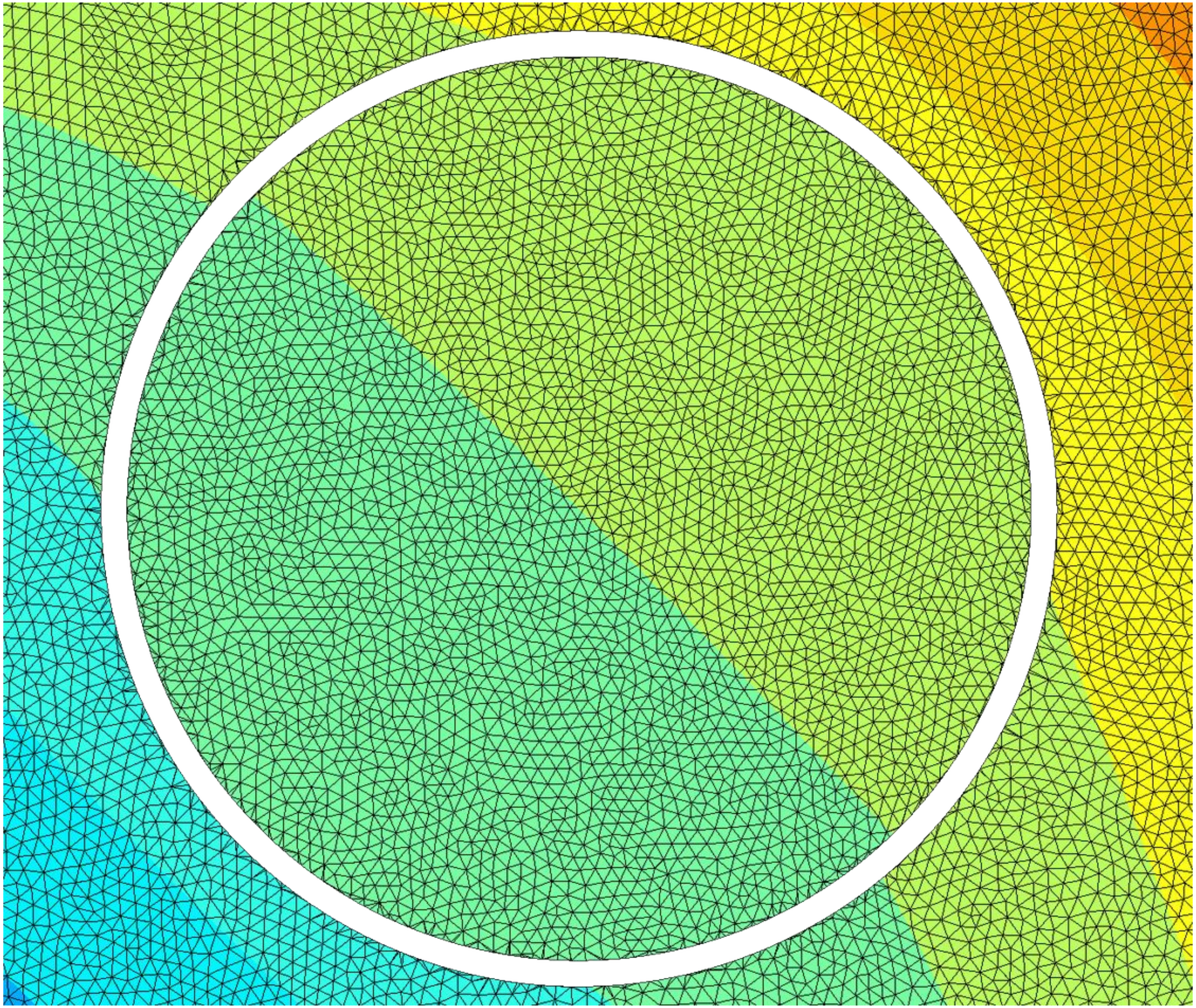


Composites : perfect glueing

Imperfect glueing

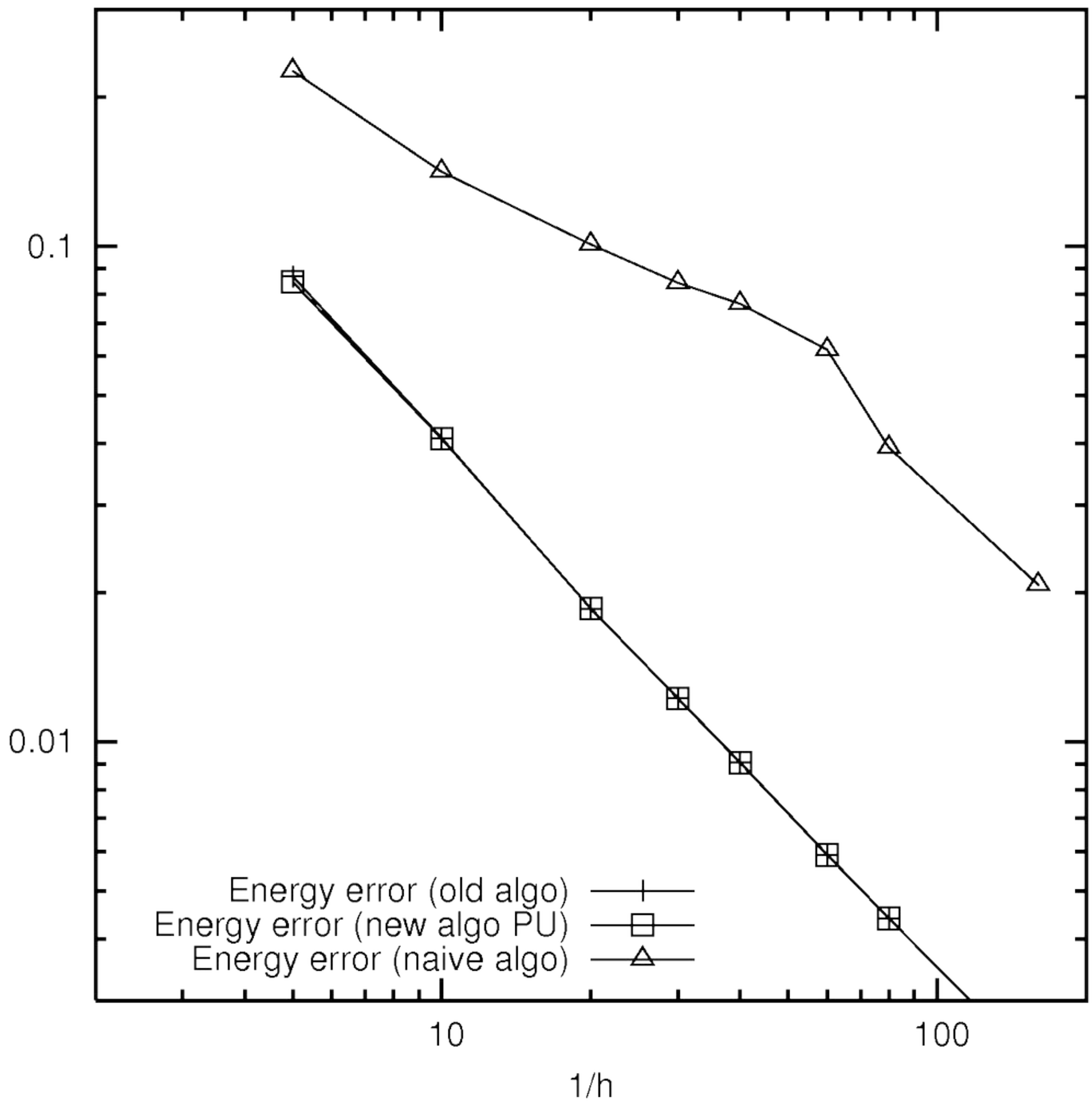
Extended Finite Elements

Dirichlet boundary conditions



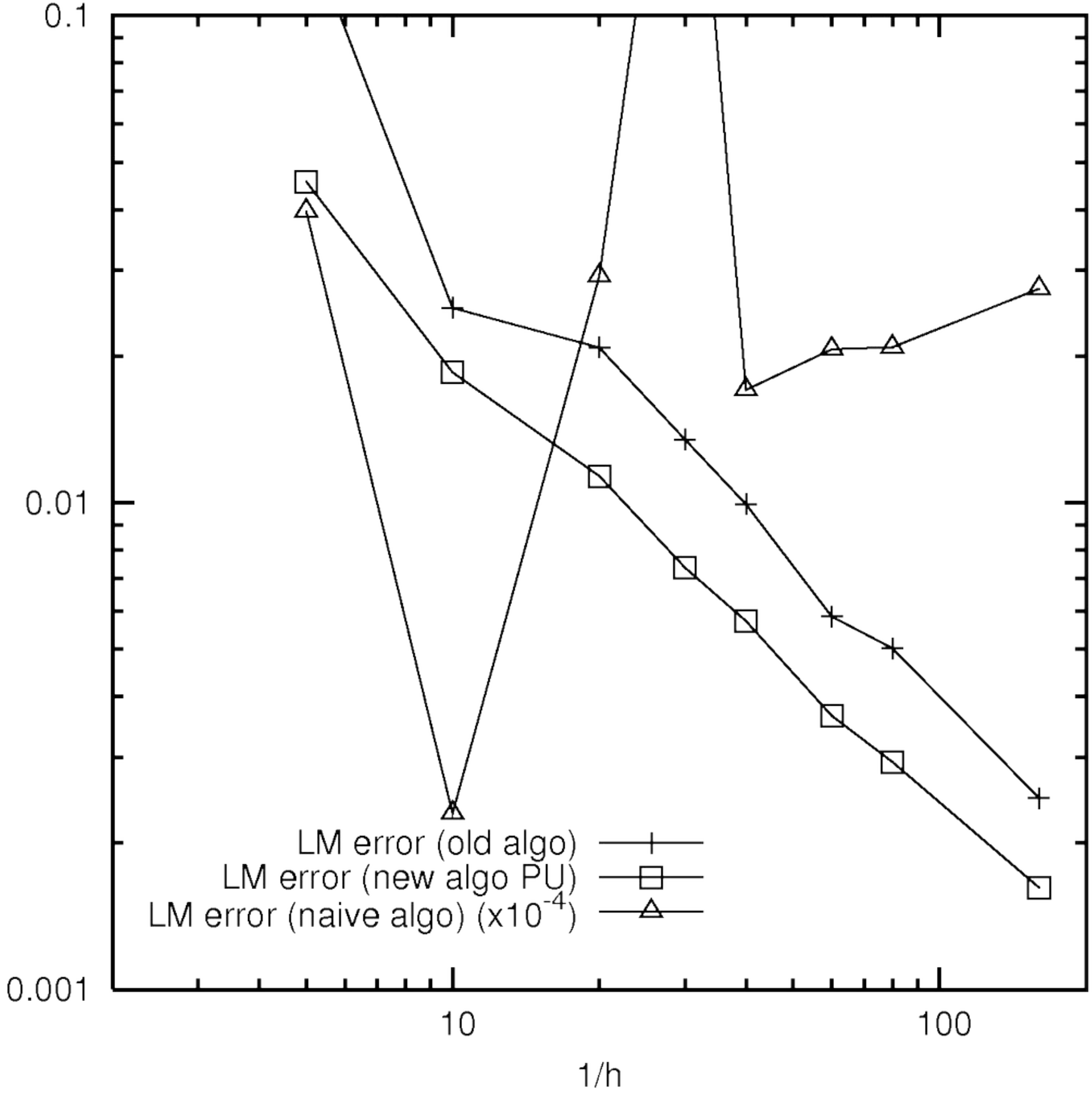
Extended Finite Elements

Dirichlet boundary conditions

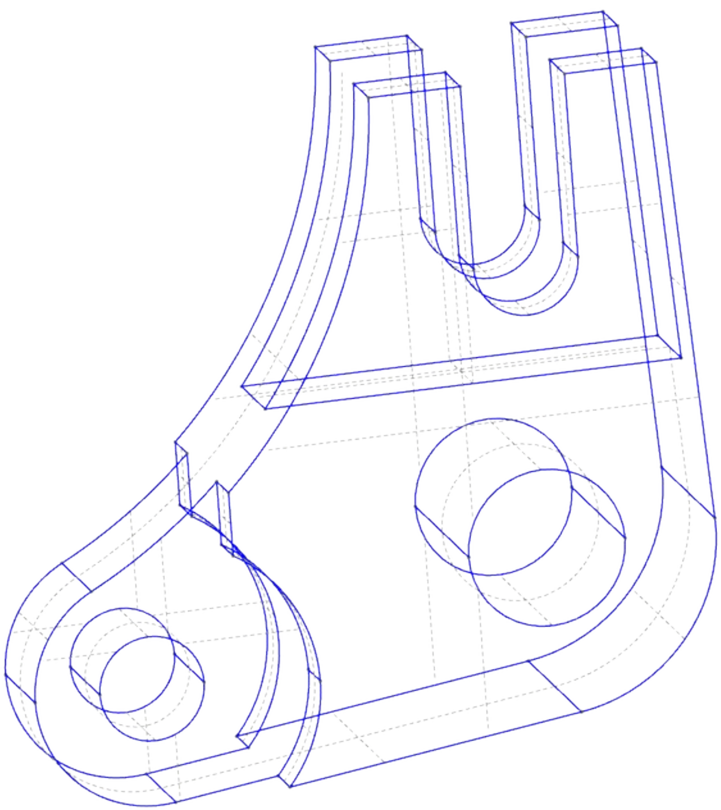


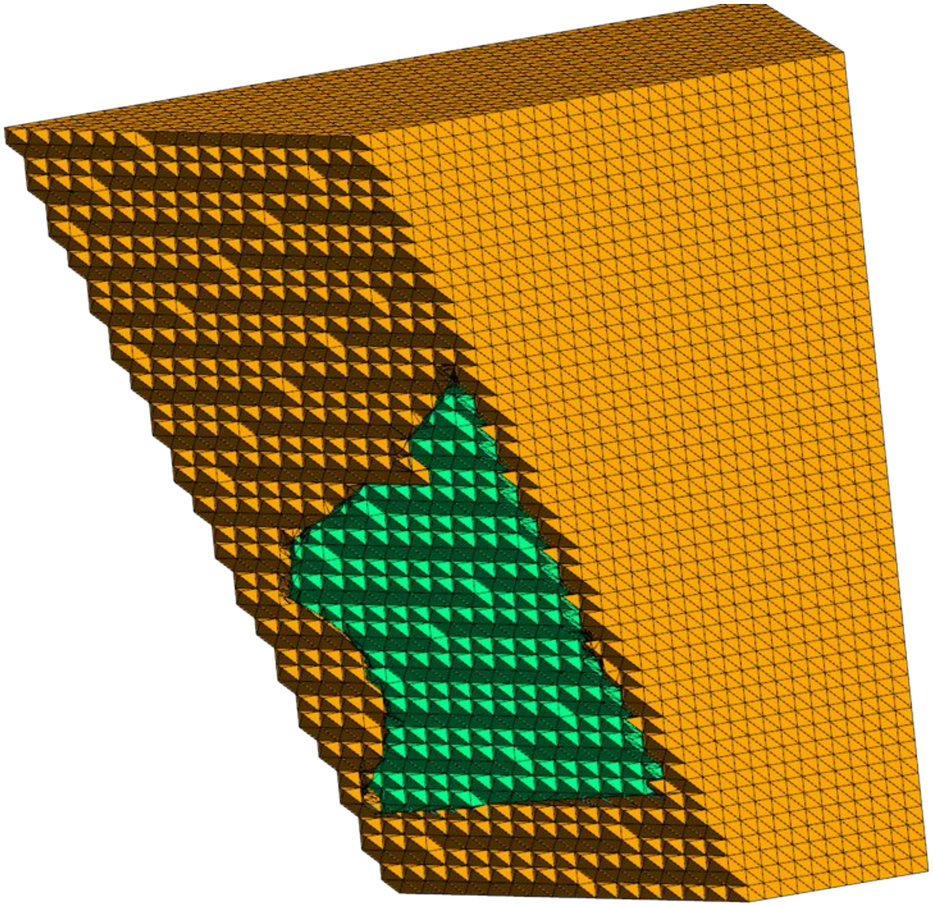
Extended Finite Elements

Dirichlet boundary conditions

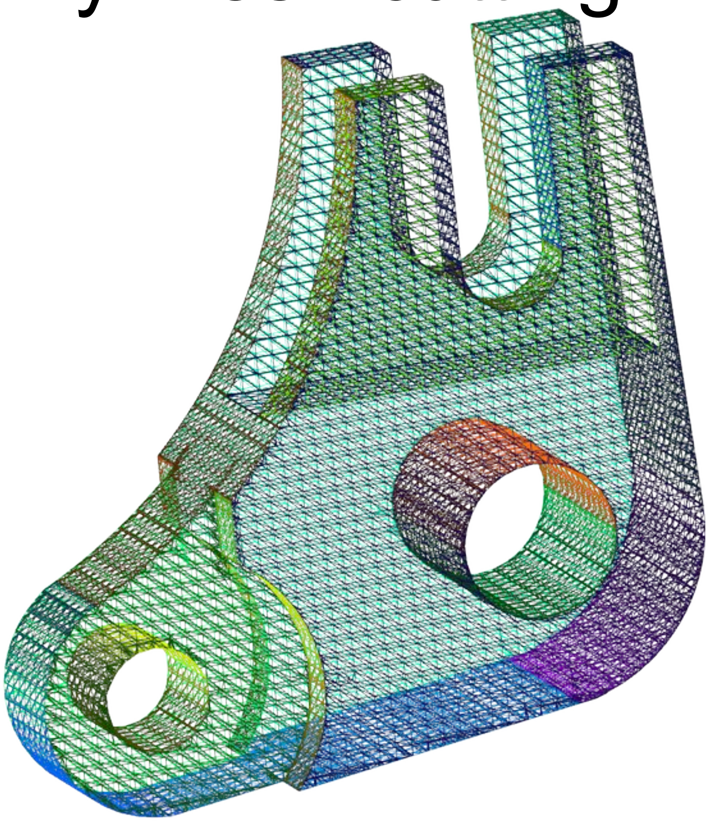


- From a traditional CAD (B-rep) representation ...





- … To an implicit representation and F.E. computation (here, no mesh generation steps, only mesh cutting …)



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Nota :

IJNME = International journal for numerical methods in engineering (Wiley)
CMAME = Computer methods in applied mechanics and engineering (Elsevier)
FEAD = Finite element in analysis and design (Elsevier)
JCP = Journal of computational physics (Elsevier)