CAD Surfaces

CAD Surfaces

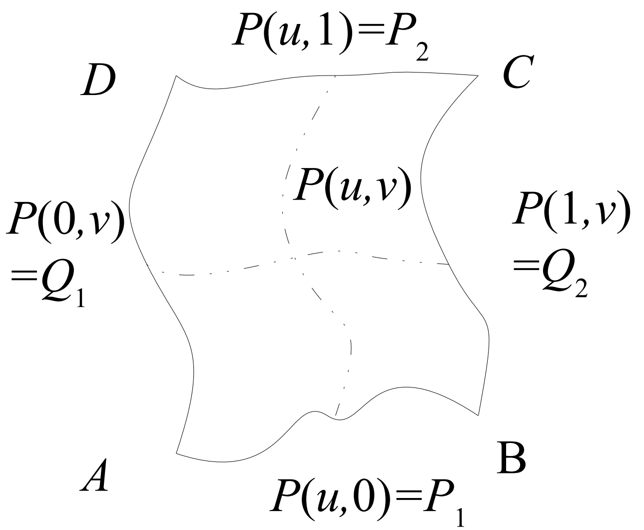
- Parametric
  - Coons patches
  - Tensor product surfaces
  - (Bézier triangle)
  - Nurbs surfaces
- Procedural (non parametric)
  - Subdivision surfaces

Coons patches

Coons patches

- Bilinear Coons patch
  - Steven Anson Coons – (published in 1967 but idea came from research done during WWII in aeronautics)
  - Let 4 parametric curves (Bézier or B-Splines or other) passing trough 4 points (A,B,C,D) :

$P_1(u)=P(u,0)$, $P_2(u)=P(u,1)$,
$Q_1(v)=P(0,v)$, $Q_2(v)=P(1,v)$ such as
$P(0,0)=A$ $\quad$ $P(1,0)=B$
$P(1,1)=C$ $\quad$ $P(0,1)=D$

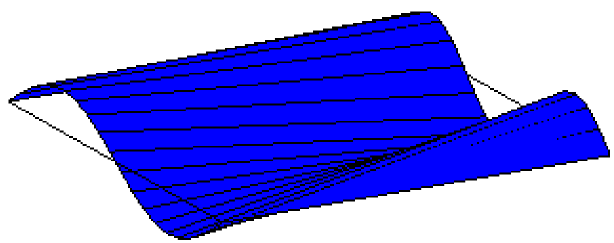


  - The surface $P(u,v)$ is carried by these 4 curves

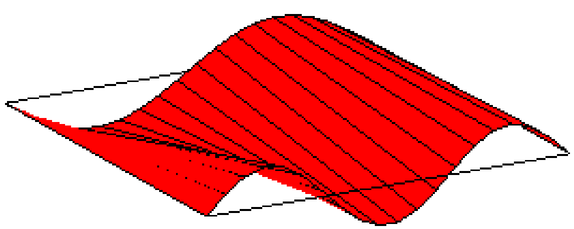
Bilinear Coons patches

- We define 3 surfaces by linear interpolation:

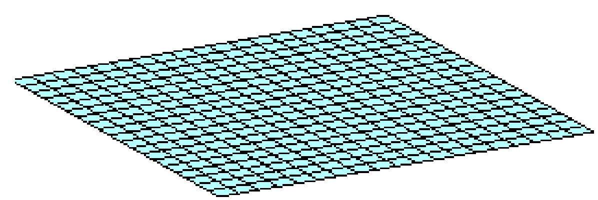
$$S_1(u,v) = (1-v)\,P(u,0) + v\,P(u,1)$$
$$S_2(u,v) = (1-u)\,P(0,v) + u\,P(1,v)$$
$$S_3(u,v) = (1-u)(1-v)\,P(0,0) + u(1-v)\,P(0,1) + v(1-u)\,P(1,0) + uvP(1,1)$$



$S_2(u,v)$



$S_1(u,v)$

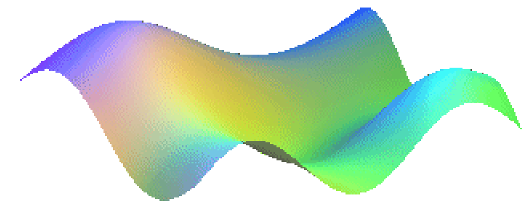


$S_3(u,v)$

Bilinear Coons patches

- The Bilinear Coons patch is defined by :

$$P(u,v) = S_1(u,v) + S_2(u,v) - S_3(u,v)$$

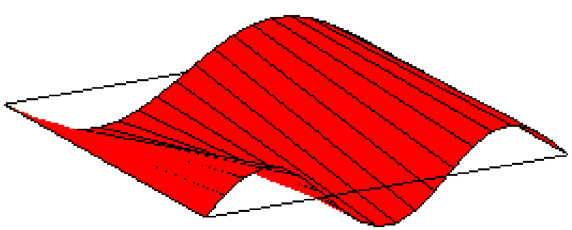


  - Why ?

  S_1 interpolates A,B,C,D and the curves P1 and P2

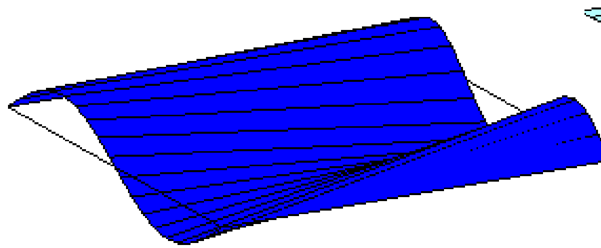
  S_2 interpolates A,B,C,D and the curves Q1 and Q2

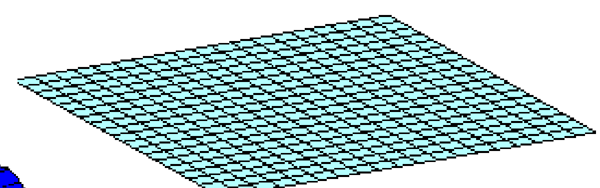
  S_1+S_2 does not interpolate A,B,C,D any more, one has to subtract a term depending on A,B,C,D and linear in u and v.



$S_1(u,v)$

$S_2(u,v)$



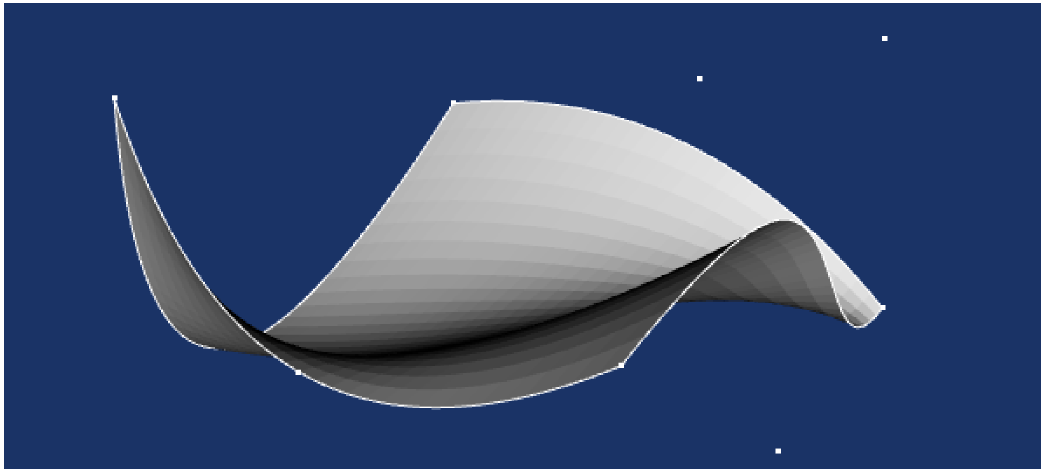


$S_3(u,v)$

Bilinear Coons patches

- Characteristics of a bilinear Coons patch
  - Easy to build
  - Based on any set of 4 boundary curves
  - However, there is no precise control of the shape of the surface “inside” the patch

e.g. it is impossible to impose a C^1 continuity between two neighbouring patches without constraints on the network of curves



Bilinear Coons patches

- Systematic notation

The surface may be expressed as :

$$P(u,v)=\begin{pmatrix}1 & F_1(u) & F_2(u)\end{pmatrix}\cdot\begin{pmatrix}0 & P(u,0) & P(u,1)\\ P(0,v) & -P(0,0) & -P(0,1)\\ P(1,v) & -P(1,0) & -P(1,1)\end{pmatrix}\cdot\begin{pmatrix}1\\ F_1(v)\\ F_2(v)\end{pmatrix}$$

- In this notation, $F_1(x)=1\text{-}x$ and $F_2(x)=x$ are **blending functions**, (x is either u or v). They can be replaced by any function to achieve e.g. a better continuity (see later).
- In the matrix, the lower right 4-by-4 square corresponds to surface S_3; the upper line to S_1 and left column to S_2.

Hermite blending

$$P(u,v)=\begin{pmatrix}1 & F_1(u) & F_2(u)\end{pmatrix}\cdot\begin{pmatrix}0 & P(u,0) & P(u,1)\\ P(0,v) & -P(0,0) & -P(0,1)\\ P(1,v) & -P(1,0) & -P(1,1)\end{pmatrix}\cdot\begin{pmatrix}1\\ F_1(v)\\ F_2(v)\end{pmatrix}$$

- It is possible to carefully select F_1 and F_2 so that they are C^1 – continuous , like e.g. Hermite polynomials :

  $$F_1(x)=2x^3-3x^2+1 \qquad F_2(x)=-2x^3+3x^2$$

- Then , as $\left.\frac{\partial F_{\{1,2\}}(x)}{\partial x}\right|_{x=\{0,1\}}=0$, the derivatives are continuous over patch boundaries.

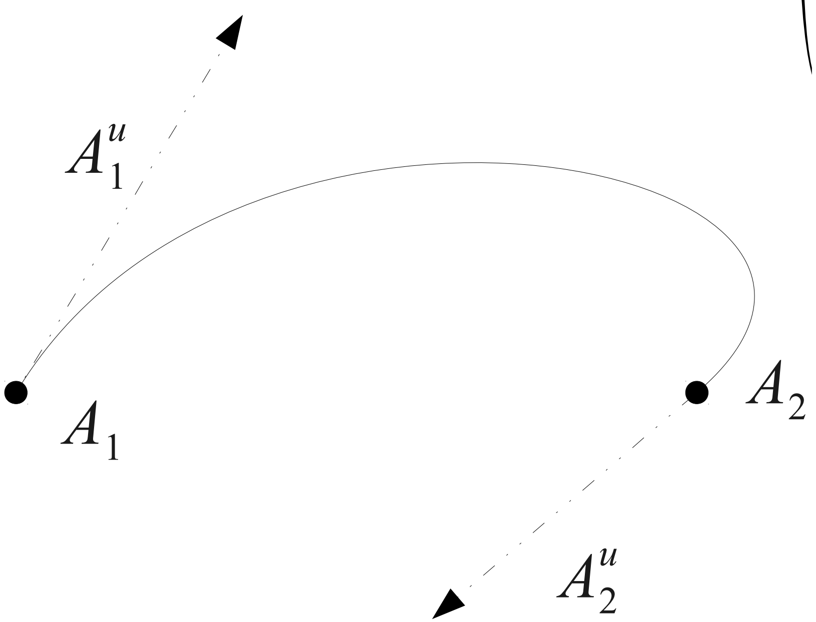
  However, this is usually not sufficient : there are too many constraints on the derivatives (they **vanish**) and it yields a surface with “flat” regions along edges and at corners.

Hermite Blending

- Hermite interpolation

  $\mathbf{H}$ is Hermite's matrix

$$C(u) = \left(A_1, A_2, A_1^u, A_2^u\right) \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u^3 \\ u^2 \\ u \\ 1 \end{pmatrix}$$



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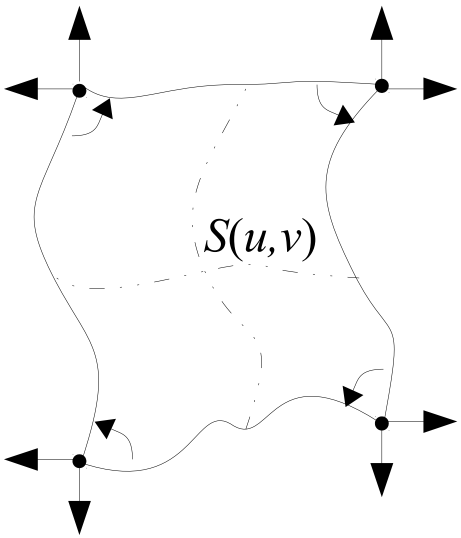
Hermite Blending

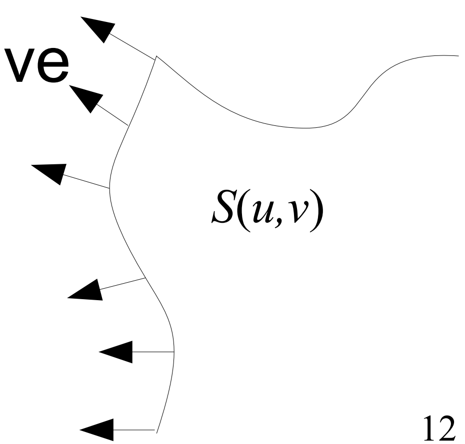
- Bicubic Hermite patch
  - Hermite interpolation in 2D
  - 4 positions at corners
  - 8 normal derivatives
  - 4 torsion vectors at corners



$$S_3(u,v)=\left(u^3,u^2,u,1\right)\mathbf{H}^T\begin{pmatrix} A_{00} & A_{01} & A^v_{00} & A^v_{01} \\ A_{10} & A_{11} & A^v_{10} & A^v_{11} \\ A^u_{00} & A^u_{01} & A^{uv}_{00} & A^{uv}_{01} \\ A^u_{10} & A^u_{11} & A^{uv}_{10} & A^{uv}_{11} \end{pmatrix}\mathbf{H}\begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

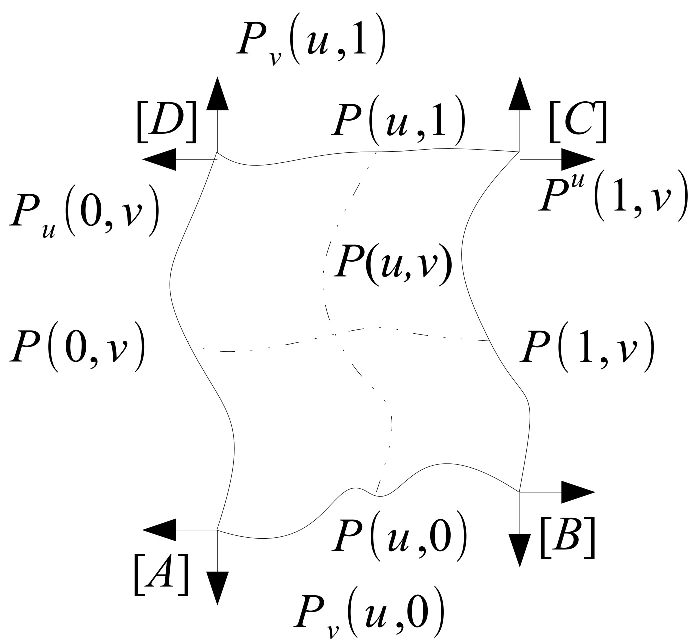
Bicubic Coons patches

- Bicubic Coons patch
  - One can build a surface that is based on any boundary curves as for the bilinear patch
  - 8 curves are necessary : 4 positional curves + 4 "curves" depicting normal derivatives
  - There are constraints between the derivative curve on one side and the positional curve on an incident side.
  - There are also constraints on the derivative curves on each corner (cross derivatives must be equal)



Bicubic Coons patches

- As for the bilinear case, we need information on boundary curves : position + derivatives, and corresponding info at the corners.



$[X] = [P(i,j)\ ,\ P(i,j)_u,\ P(i,j)_v,\ P(i,j)_{uv}]\ ,\ (i,j)=\{0,1\}^2$

Bicubic Coons patches

- Systematic notation for the bicubic Coons patch

$$P(u,v) = \begin{pmatrix} 1 & F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{pmatrix} \cdot \begin{pmatrix} 0 & P(u,0) & P(u,1) & P_v(u,0) & P_v(u,1) \\ P(0,v) & -P(0,0) & -P(0,1) & -P_v(0,0) & -P_v(0,1) \\ P(1,v) & -P(1,0) & -P(1,1) & -P_v(1,0) & -P_v(1,1) \\ P_u(0,v) & -P_u(0,0) & -P_u(0,1) & -P_{uv}(0,0) & -P_{uv}(0,1) \\ P_u(1,v) & -P_u(1,0) & -P_u(1,1) & -P_{uv}(1,0) & -P_{uv}(1,1) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ F_1(v) \\ F_2(v) \\ F_3(v) \\ F_4(v) \end{pmatrix}$$

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Bicubic Coons patches

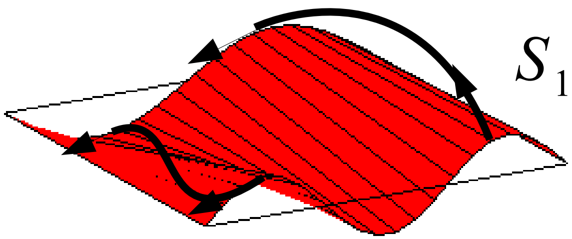
$$= \begin{pmatrix} 1 & u^3 & u^2 & u & 1 \end{pmatrix} \cdot \left(\mathbf{H}^*\right)^{\mathrm{T}}$$

$$\cdot \begin{pmatrix} 0 & P(u,0) & P(u,1) & P_v(u,0) & P_v(u,1) \\ P(0,v) & -P(0,0) & -P(0,1) & -P_v(0,0) & -P_v(0,1) \\ P(1,v) & -P(1,0) & -P(1,1) & -P_v(1,0) & -P_v(1,1) \\ P_u(0,v) & -P_u(0,0) & -P_u(0,1) & -P_{uv}(0,0) & -P_{uv}(0,1) \\ P_u(1,v) & -P_u(1,0) & -P_u(1,1) & -P_{uv}(1,0) & -P_{uv}(1,1) \end{pmatrix} \cdot \mathbf{H}^* \cdot \begin{pmatrix} 1 \\ v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

… with $\mathbf{H}^* = \begin{pmatrix} 1 & \cdots 0 \cdots \\ \vdots & \\ 0 & \mathbf{H} \\ \vdots & \end{pmatrix}$ and $\mathbf{H} = \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

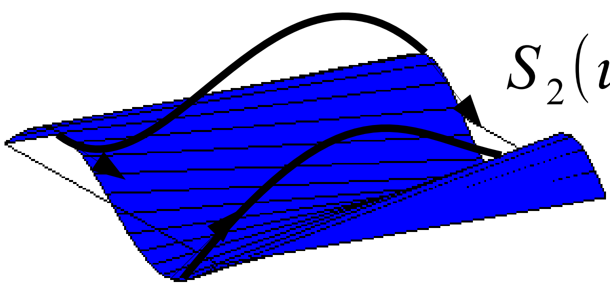
Bicubic Coons patches

- As for bilinear patches, it can be decomposed into three surfaces



$S_1(u,v)$

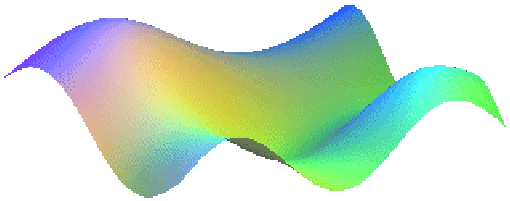
$$S_1(u,v)=\left(P(u,0),\,P(u,1),\,P_v(u,0),\,P_v(u,1)\right)\mathbf{H}\begin{pmatrix}v^3\\v^2\\v\\1\end{pmatrix}$$



$S_2(u,v)$

$$S_2(u,v)=\left(P(0,v),\,P(1,v),\,P_u(0,v),\,P_u(1,v)\right)\mathbf{H}\begin{pmatrix}u^3\\u^2\\u\\1\end{pmatrix}$$

Bicubic Coons patches



$$S_3(u,v)=\left(u^3, u^2, u, 1\right)\cdot \mathbf{H}^{\mathrm{T}} \cdot \begin{pmatrix} P(0,0) & P(0,1) & P_v(0,0) & P_v(0,1) \\ P(1,0) & P(1,1) & P_v(1,0) & P_v(1,1) \\ P_u(0,0) & P_u(0,1) & P_{uv}(0,0) & P_{uv}(0,1) \\ P_u(1,0) & P_u(1,1) & P_{uv}(1,0) & P_{uv}(1,1) \end{pmatrix} \cdot \mathbf{H} \cdot \begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

$$P(u,v)=S_1(u,v)+S_2(u,v)-S_3(u,v)$$

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Bicubic Coons patches

- Bicubic Coons patch

$$P(u,v)=S_1(u,v)+S_2(u,v)-S_3(u,v)$$

$$S_3(u,v)=\left(u^3,u^2,u,1\right)\mathbf{H}^T\begin{pmatrix} A_{00} & A_{01} & A_{00}^v & A_{01}^v \\ A_{10} & A_{11} & A_{10}^v & A_{11}^v \\ A_{00}^u & A_{01}^u & A_{00}^{uv} & A_{01}^{uv} \\ A_{10}^u & A_{11}^u & A_{10}^{uv} & A_{11}^{uv} \end{pmatrix}\mathbf{H}\begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

The terms of the matrix are computed with the help of boundary curves and verify the following conditions :

$$A_{00}=P(0,0) \quad A_{01}=P(0,1) \quad A_{00}^v=P_v(0,0) \quad A_{01}^v=P_v(0,1)$$
$$A_{10}=P(1,0) \quad A_{11}=P(1,1) \quad A_{10}^v=P_v(1,0) \quad A_{11}^v=P_v(1,1)$$
$$A_{00}^u=P_u(0,0) \quad A_{01}^u=P_u(0,1) \quad A_{00}^{uv}=P_{uv}(0,0) \quad A_{01}^{uv}=P_{uv}(0,1)$$
$$A_{10}^u=P_u(1,0) \quad A_{11}^u=P_u(1,1) \quad A_{10}^{uv}=P_{uv}(1,0) \quad A_{11}^{uv}=P_{uv}(1,1)$$

Bicubic Ferguson patch

- Ferguson patch

$$P(u,v)=S_1(u,v)+S_2(u,v)-S_3(u,v)$$

$$S_3(u,v)=\left(u^3,u^2,u,1\right)\mathbf{H}^T\begin{pmatrix}A_{00} & A_{01} & A_{00}^v & A_{01}^v\\ A_{10} & A_{11} & A_{10}^v & A_{11}^v\\ A_{00}^u & A_{01}^u & A_{00}^{uv} & A_{01}^{uv}\\ A_{10}^u & A_{11}^u & A_{10}^{uv} & A_{11}^{uv}\end{pmatrix}\mathbf{H}\begin{pmatrix}v^3\\ v^2\\ v\\ 1\end{pmatrix}$$

The terms of the matrix are computed with the help of boundary curves and verify the following conditions :

$A_{00}=P(0,0)$ $\quad A_{01}=P(0,1)$ $\quad A_{00}^v=P_v(0,0)$ $\quad A_{01}^v=P_v(0,1)$

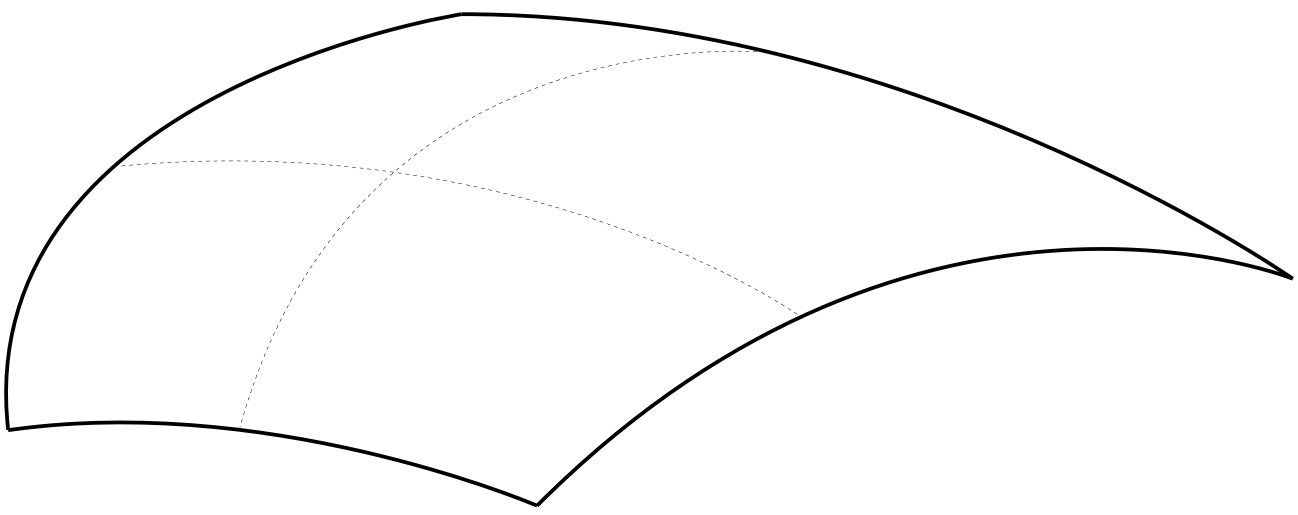
$A_{10}=P(1,0)$ $\quad A_{11}=P(1,1)$ $\quad A_{10}^v=P_v(1,0)$ $\quad A_{11}^v=P_v(1,1)$

$A_{00}^u=P_u(0,0)$ $\quad A_{01}^u=P_u(0,1)$ $\quad A_{00}^{uv}$ such that $\dfrac{\partial^2 P}{\partial u\partial v}(0,0)=0$ $\quad A_{01}^{uv}$ such that $\dfrac{\partial^2 P}{\partial u\partial v}(0,1)=0$

$A_{10}^u=P_u(1,0)$ $\quad A_{11}^u=P_u(1,1)$ $\quad A_{10}^{uv}$ such that $\dfrac{\partial^2 P}{\partial u\partial v}(1,0)=0$ $\quad A_{11}^{uv}$ such that $\dfrac{\partial^2 P}{\partial u\partial v}(1,1)=0$

Bicubic Coons patches

- We can impose the position and the normal tangent along the boundaries
- Remain the problem of the continuity at every corner
  - We usually impose that cross derivatives vanish → Ferguson patch
  - Other constraints may be found in the literature...



“Tensor product” surfaces

Tensor product surfaces

- Parametric surfaces as a polar form

$$S(u,v)=\sum_k N_k(u,v)\,P_k$$

  - One shape function per control point
- If N_k is separable : « Tensor product » surface
  - Combination of elementary curves/shape functions independently defined on u and v.

$$S(u,v)=\sum_i \sum_j G_i(u)\,H_j(v)\,P_{ij}$$

    Usually built upon Bézier and B-Splines curves/SFs
  - The “unique” shape function is $N_k(u,v)=G_i(u)\,H_j(v)$

B-Spline surfaces

- B-Splines surfaces uses 1D B-Spline shape fns.
  - Definition as tensor product :

$$S(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m} N_i^p(u)\,N_j^q(v)\,P_{ij}$$

  - Every variable u and v has a degree (p and q) and a nodal sequence U and V :

$$U=\{\underbrace{0,\cdots,0}_{p+1},u_{p+1},\cdots,u_{r-p-1},\underbrace{1,\cdots,1}_{p+1}\} \qquad (r+1 \text{ nodes})$$

$$V=\{\underbrace{0,\cdots,0}_{q+1},v_{q+1},\cdots,v_{s-q-1},\underbrace{1,\cdots,1}_{q+1}\} \qquad (s+1 \text{ nodes})$$

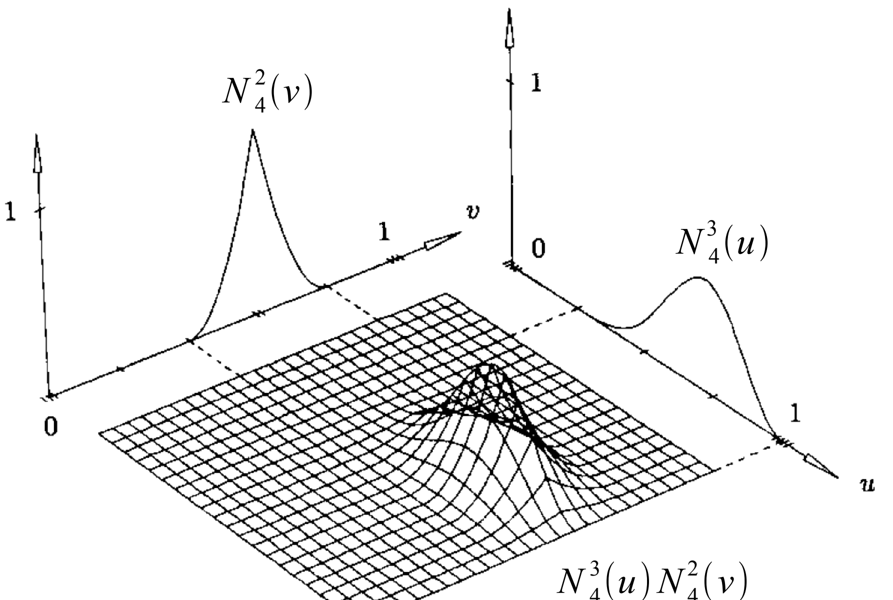
  - The control points forms a regular net P_{ij} ($n+1$ times $m+1$) values.
  - We have the following relations : $r=n+p+1 \quad s=m+q+1$

B-Spline surfaces

- Example of basis functions

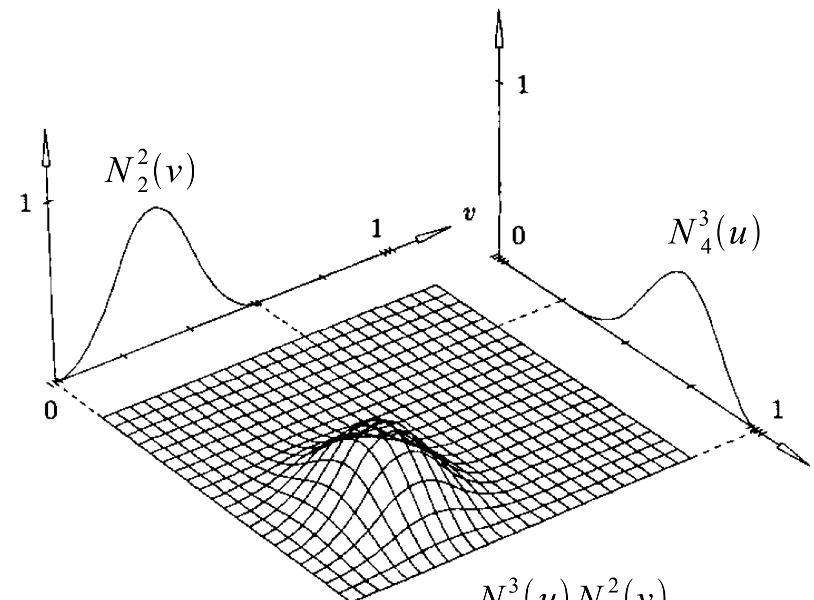
$$U=\{0,0,0,0,1/4,1/2,3/4,1,1,1,1\} \qquad p=3$$

$$V=\{0,0,0,1/5,2/5,3/5,3/5,4/5,1,1,1\} \qquad q=2$$



$N_4^3(u)N_4^2(v)$

Basis function associated to P_{44}



$N_4^3(u)N_2^2(v)$

Basis function associated to P_{42}

L. Piegl « The NURBS Book »

B-Spline surfaces

- Properties of surface basis functions
  - Extrema
    If $p>0$ and $q>0$, $N_i^p(u)\,N_j^q(v)$ has a unique maximum.
  - Continuity
    Inside rectangles formed by the nodes u_i and v_j, the SF are infinitely differentiable.
    At a node u_i (resp. v_j), $N_i^p(u)\,N_j^q(v)$ is $(p$-$k)$ (resp. $(q$-$k)$) times differentiable, k being the node multiplicity u_i (resp. v_j)
    The continuity with respect to u (resp. v) depends solely on the nodal sequence U (resp. V).

- Properties of surface basis functions
  - A consequence of properties of the 1D shape functions
  - Non-negativity
    $N_i^p(u)\,N_j^q(v)\geq 0\ \forall\, i, j, p, q, u, v$
  - Partition of unity
    $$\sum_{i=0}^{n}\sum_{j=0}^{m} N_i^p(u)\,N_j^q(v)=1\ \forall\,(u,v)\in[u_{min}, u_{max}]\times[v_{min}, v_{max}]$$
  - Compact support
    $N_i^p(u)\,N_j^q(v)=0 \text{ outside } (u,v)\in[u_i, u_{i+p+1}[\times[v_j, v_{j+q+1}[$
    There are at most $(p+1)(q+1)$ non zero SF in a given interval $[u_{i_0}, u_{i_0+1}[\times[v_{j_0}, v_{j_0+1}[$.
    In particular $N_i^p(u)\,N_j^q(v)\neq 0 \quad i_0-p\leq i\leq i_0 \quad j_0-q\leq j\leq j_0$

B-Spline surfaces

- Properties of surface basis functions
  - Extrema
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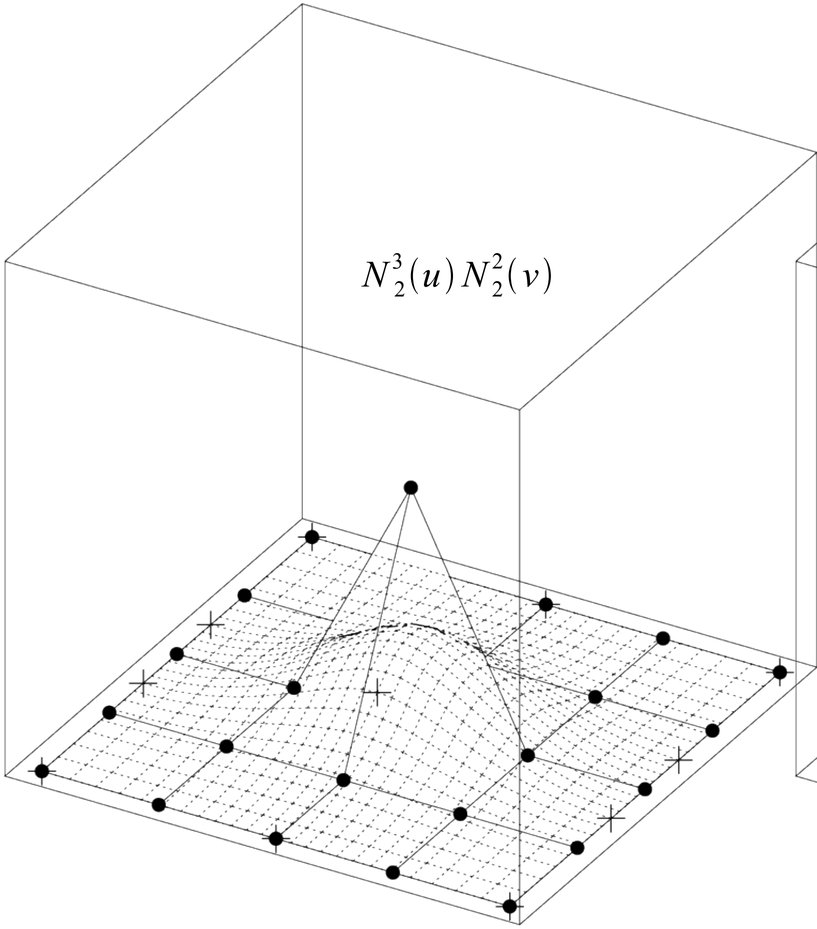
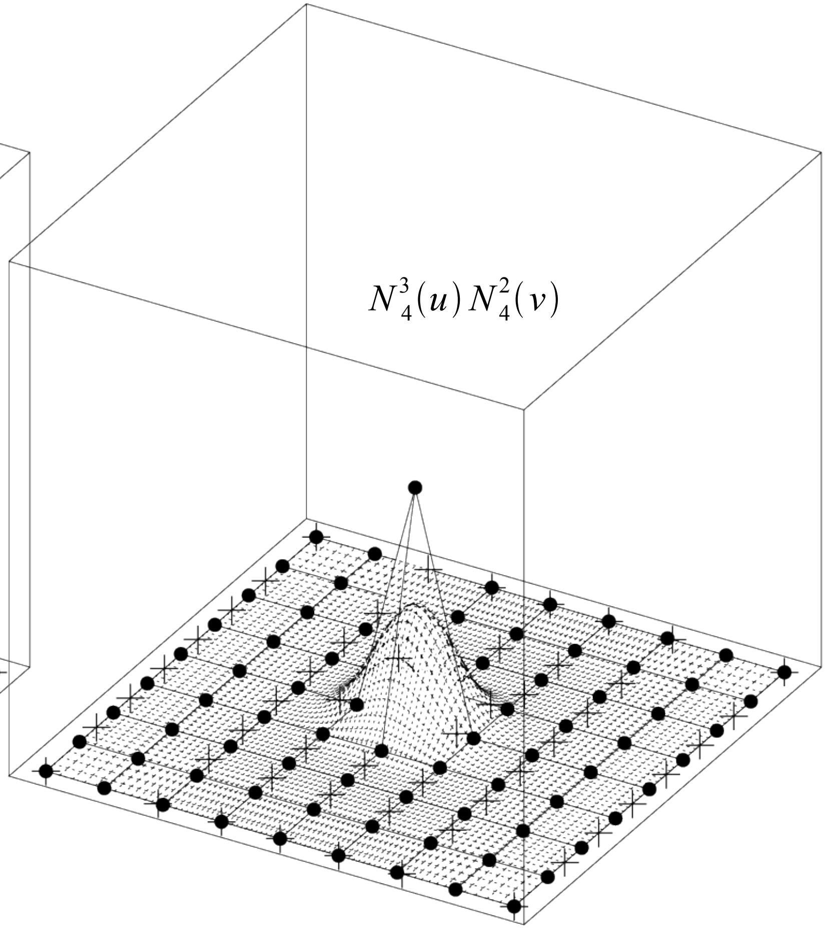
B-Spline surfaces





$$U=\{0,0,0,0,1,2,2,2,2\} \quad p=3 \quad U=\{0,0,0,0,1,2,3,4,5,6,6,6,6\}$$

$$V=\{0,0,0,1,2,3,3,3\} \quad q=2 \quad V=\{0,0,0,1,2,3,4,5,6,7,7,7\}$$

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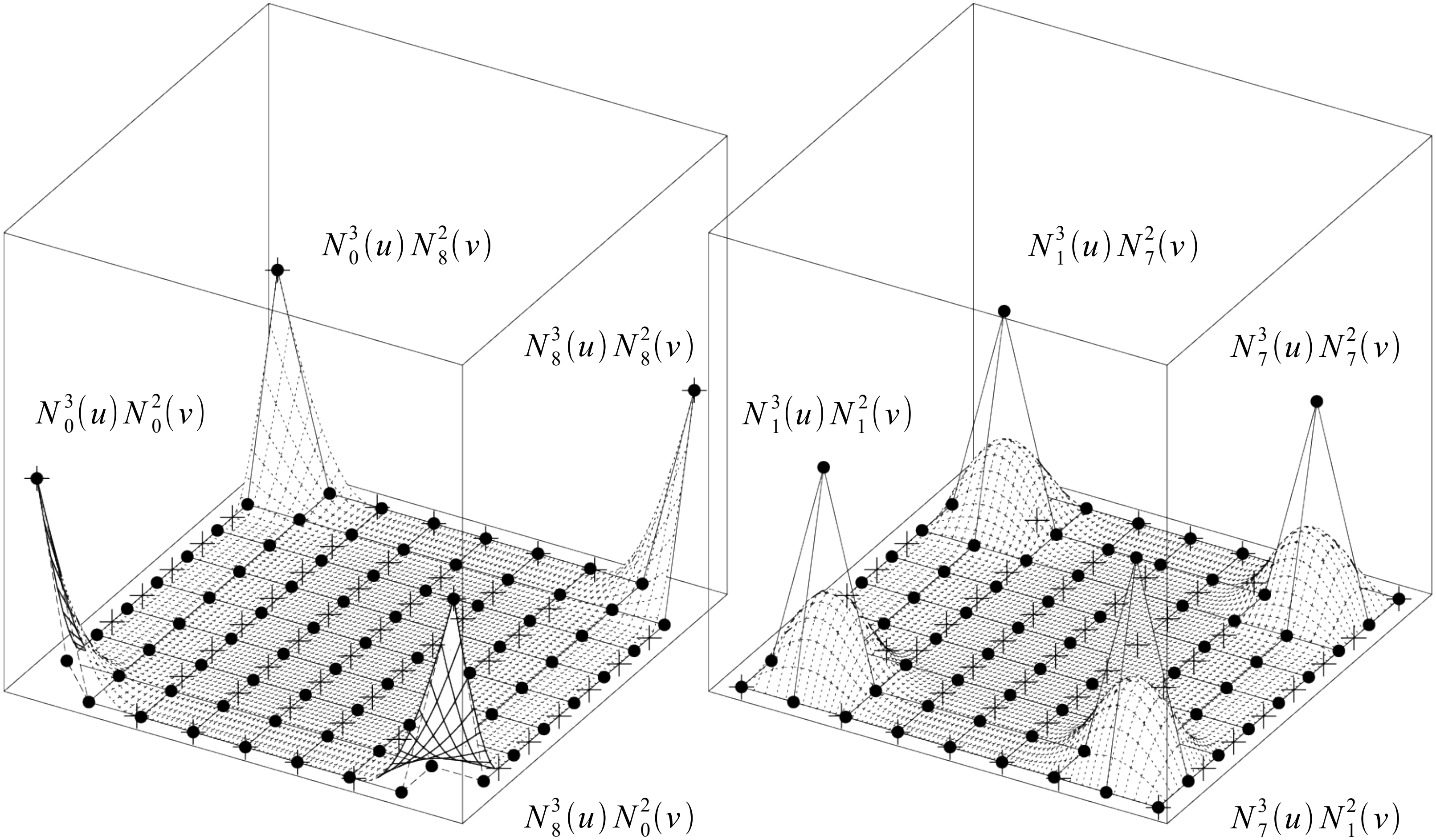
B-Spline surfaces



$$U=\{0,0,0,0,1,2,3\ ,4,5,6,6,6,6\} \qquad p=3$$
$$V=\{0,0,0,1,2,3,4,5,6,7,7,7\} \qquad q=2$$

- Computation of a point on the surface

  1 – Find the nodal interval in which u is located

  $$u \in [u_i, u_{i+1}[$$

  2 – Compute the non vanishing 1D shape functions

  $$N_{i-p}^{p}(u), \cdots, N_{i}^{p}(u)$$

  3 – Find the nodal interval in which v is located

  $$v \in [v_j, v_{j+1}[$$

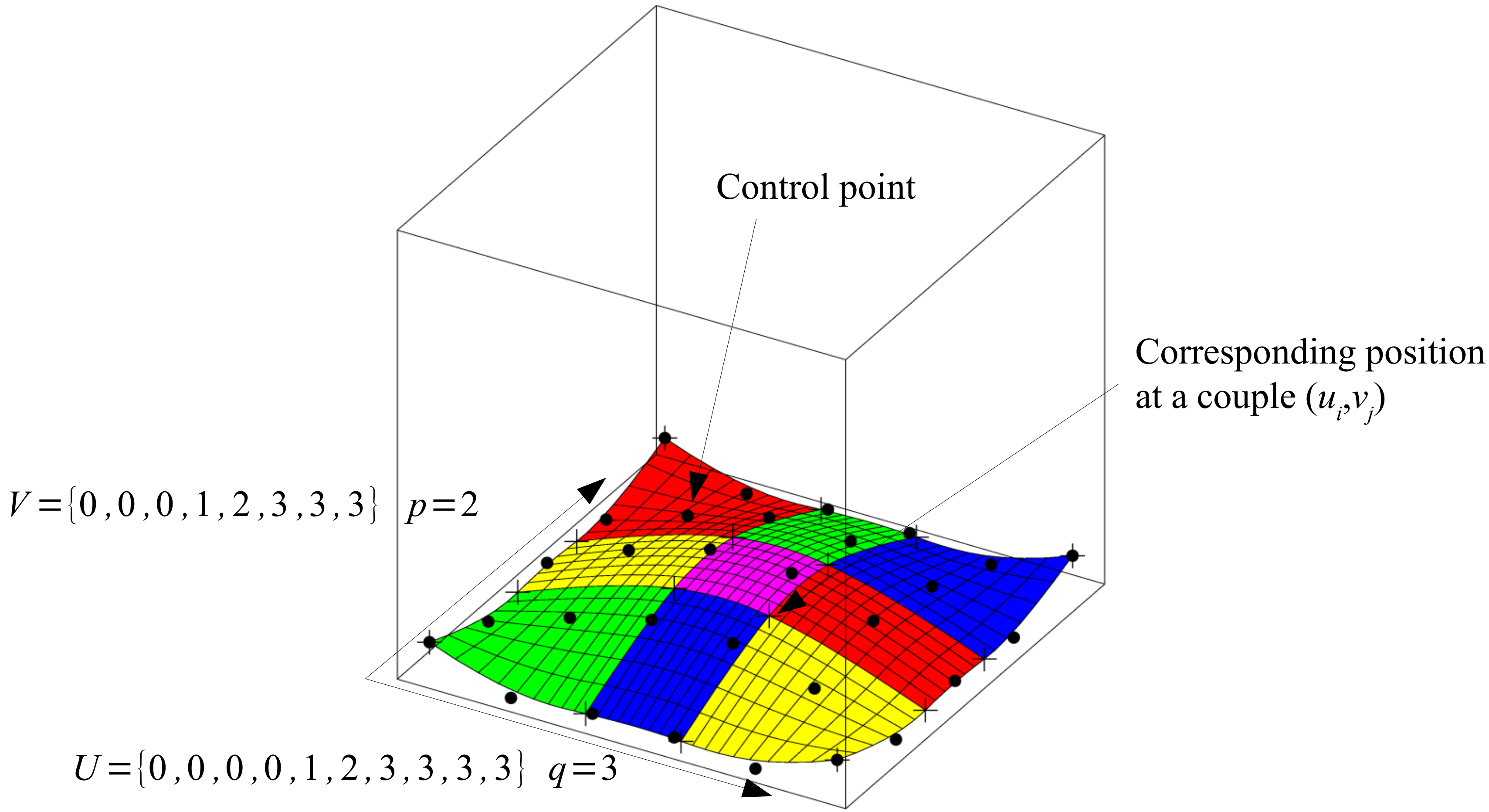
  4 – Compute the non vanishing 1D shape functions

  $$N_{j-q}^{q}(v), \cdots, N_{j}^{q}(v)$$

  5 – Multiply the SFs with the adequate control points

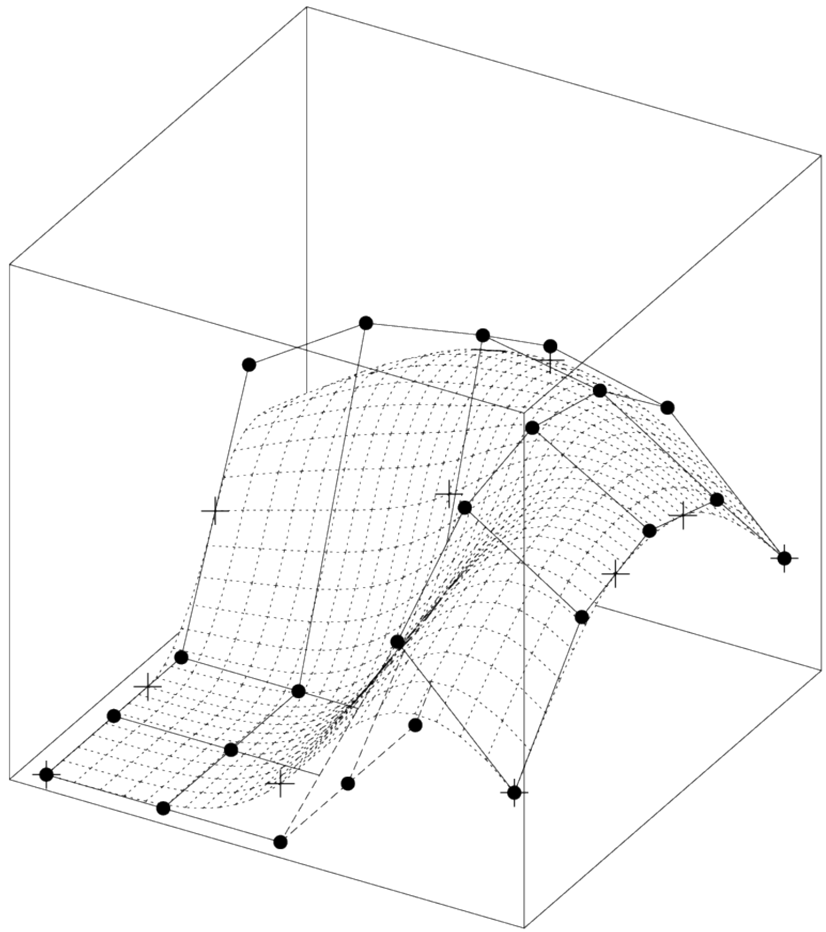
$$S(u,v) = \sum_{k} \sum_{l} N_{k}^{p}(u) P_{kl} N_{l}^{q}(v) \qquad i-p \leq k \leq i \quad , \quad j-q \leq l \leq j$$

B-Spline surfaces



- Each coloured square has an independent polynomial expression

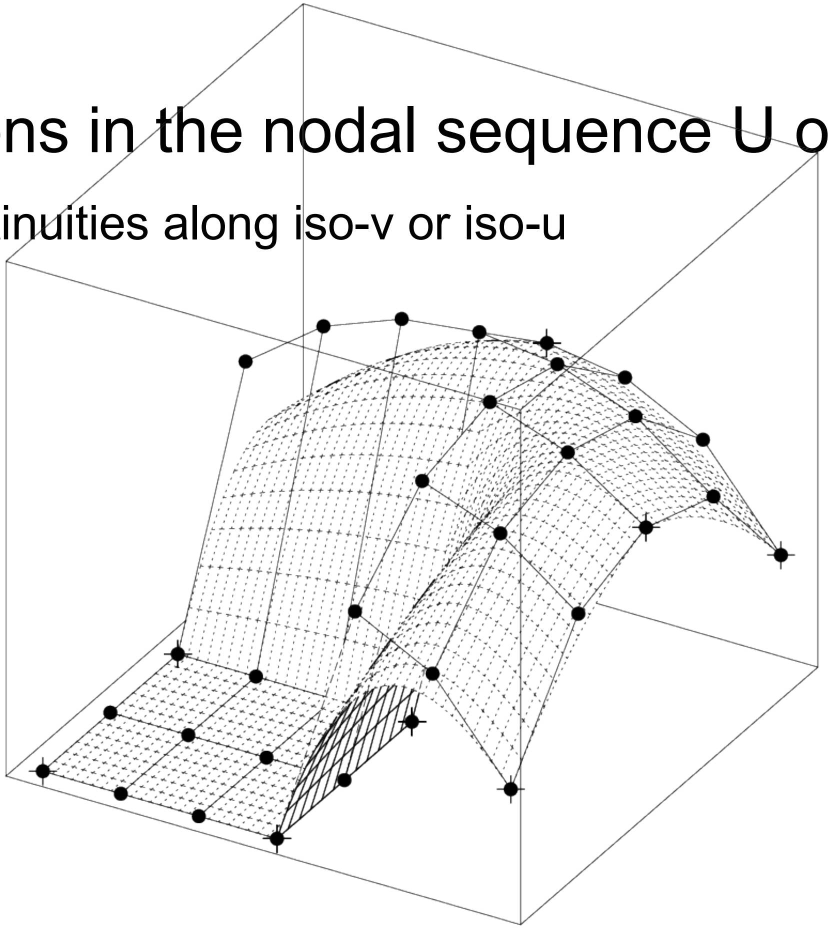
B-Spline surfaces



$$U=\{0,0,0,0,1,2,2,2,2\} \qquad p=3$$
$$V=\{0,0,0,1,2,3,3,3\} \qquad q=2$$

B-Spline surfaces

- Repetitions in the nodal sequence U or V
  - Discontinuities along iso-v or iso-u



$$U=\{0,0,0,0,2,2,2,4,4,4,4\} \qquad p=3$$
$$V=\{0,0,0,1.5,1.5,3,3,3\} \qquad q=2$$

- Properties of the B-Spline surface
  - Interpolate the 4 corners if the nodal sequences are of the form

$$U=\{\underbrace{0,\cdots,0}_{p+1},u_{p+1},\cdots,u_{r-p-1},\underbrace{1,\cdots,1}_{p+1}\}$$

$$V=\{\underbrace{0,\cdots,0}_{q+1},v_{q+1},\cdots,v_{s-q-1},\underbrace{1,\cdots,1}_{q+1}\}$$

  - If the nodal sequences correspond to Bézier curves :

$$U=\{\underbrace{0,\cdots,0}_{p+1},\underbrace{1,\cdots,1}_{p+1}\} \qquad V=\{\underbrace{0,\cdots,0}_{q+1},\underbrace{1,\cdots,1}_{q+1}\}$$

then the surface is called a Bézier patch.

- Properties of the B-Spline surface
  - The surface has the property of affine invariance (invariance by translation in particular)
  - The convex hull of the control points contains the surface.
  - In every interval $(u,v)\in[u_{i_0},u_{i_0+1}[\times[v_{j_0},v_{j_0+1}[$, the portion of the surface is in the convex hull of the control points P_{ij} , (i,j) such that $i_0-p\leq i\leq i_0 \quad j_0-q\leq j\leq j_0$
  - Control points may have a local control
  - There is **no** variation diminishing property (on the contrary to B-Spline/Bézier curves)

- Isoparametrics
  - Computation of isoparametrics is easy :
    Set $u=u_0$

$$C_{u_0}(v)=S(u_{0,}v)=\sum_{i=0}^{n}\sum_{j=0}^{m}N_i^p(u_0)\,N_j^q(v)\,P_{ij}$$

$$=\sum_{j=0}^{m}N_j^q(v)\sum_{i=0}^{n}N_i^p(u_0)\,P_{ij}=\sum_{j=0}^{m}N_j^q(v)\,Q_j(u_0)$$

$$\text{with } Q_j(u_0)=\sum_{i=0}^{n}N_i^p(u_0)\,P_{ij}$$

  - same with $v=v_0$

$$C_{v_0}(u)=S(u\,,v_0)=\sum_{i=0}^{n}N_i^p(u)\,Q_i(v_0)$$

$$\text{with } Q_i(v_0)=\sum_{j=0}^{m}N_j^q(v_0)\,P_{ij}$$

- Derivatives of a B-Spline surface
  - We want to compute $\frac{\partial^{k+l}}{\partial u^k \partial v^l} S(u,v)$
  - Differentiation of basis functions :

$$\frac{\partial^{k+l}}{\partial u^k \partial v^l} S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{\partial^{k+l}}{\partial u^k \partial v^l} N_i^p(u) N_j^q(v) P_{ij}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{\partial^k}{\partial u^k} N_i^p(u) \frac{\partial^l}{\partial v^l} N_j^q(v) P_{ij}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} N_i^{p(k)}(u) N_j^{q(l)}(v) P_{ij}$$

B-Spline surfaces

- Derivatives expressed as B-Spline surfaces
  - Let's derive formally S with respect to u:

$$\frac{\partial S(u,v)}{\partial u}=\sum_{j=0}^{m} N_j^q(v)\left(\frac{\partial}{\partial u}\sum_{i=0}^{n} N_i^p(u)P_{ij}\right)$$

$$=\sum_{j=0}^{m} N_j^q(v)\left(\frac{\partial}{\partial u}C_j(u)\right)$$

$$\text{with } C_j(u)=\sum_{i=0}^{n} N_i^p(u)P_{ij}$$

  - We want to apply equations seen for curves :

B-Spline surfaces

- Derivatives of the curve $C_j(u)=\sum_{i=0}^{n} N_i^p(u)\,P_{ij}$

$$U=\{\underbrace{u_0,\cdots,u_p}_{p+1\ \text{times}},\cdots,\underbrace{u_{m-p},\cdots,u_m}_{p+1\ \text{times}}\}$$

$$U'=\{\underbrace{u'_0,\cdots,u'_{p-1}}_{p\ \text{times}},\cdots,\underbrace{u'_{m-p-1},\cdots,u'_{m-2}}_{p\ \text{times}}\}\ \text{with}\ u'_i=u_{i+1}$$

$$P'(u)=\sum_{i=0}^{n-1} N_i^{p-1}(u)\,Q_i \quad \text{with} \quad N_i^{p-1}\ \text{defined on}\ U'$$

$$Q_i=p\,\frac{P_{i+1\,j}-P_{ij}}{u_{i+d+1}-u_{i+1}}$$

- We obtain :

$$\frac{\partial S(u,v)}{\partial u}=\sum_{i=0}^{n-1}\sum_{j=0}^{m} N_i^{p-1}(u)\,N_j^q(v)\,P_{ij}^{(1,0)}$$

$$\text{with}\quad P_{ij}^{(1,0)}=p\frac{P_{i+1\,j}-P_{ij}}{u_{i+p+1}-u_{i+1}}$$

$$U=\{\underbrace{u_0,\cdots,u_p}_{p+1\ \text{times}},\cdots,\underbrace{u_{m-p},\cdots,u_m}_{p+1\ \text{times}}\}$$

$$U^{(1)}=\{\underbrace{u_0^{(1)},\cdots,u_{p-1}^{(1)}}_{p\ \text{times}},\cdots,\underbrace{u_{m-p-1}^{(1)},\cdots,u_{m-2}^{(1)}}_{p\ \text{times}}\}\ \text{with}\ u_i^{(1)}=u_{i+1},\ 0\le i\le m-2$$

B-Spline surfaces

- Let's derive formally S with respect to v:

$$\frac{\partial S(u,v)}{\partial v} = \sum_{i=0}^{n} \sum_{j=0}^{m-1} N_i^p(u) N_j^{q-1}(v) P_{ij}^{(0,1)}$$

$$\text{with} \quad P_{ij}^{(0,1)} = q \frac{P_{i\,j+1} - P_{ij}}{v_{j+q+1} - v_{j+1}}$$

$$V = \{\underbrace{v_0, \cdots, v_q}_{q+1 \text{ times}}, \cdots, \underbrace{v_{n-q}, \cdots, v_n}_{q+1 \text{ times}}\}$$

$$V^{(1)} = \{\underbrace{v_0^{(1)}, \cdots, v_{q-1}^{(1)}}_{q \text{ times}}, \cdots, \underbrace{v_{n-q-1}^{(1)}, \cdots, v_{n-2}^{(1)}}_{q \text{ times}}\} \text{ with } v_j^{(1)} = v_{j+1}, 0 \le j \le n-2$$

- Let's derive formally S with respect to u , then v:

$$\frac{\partial^2 S(u,v)}{\partial u \partial v} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} N_i^{p-1}(u) N_j^{q-1}(v) P_{ij}^{(1,1)}$$

with $P_{ij}^{(1,1)} = q \dfrac{P_{i\,j+1}^{(1,0)} - P_{ij}^{(1,0)}}{v_{j+q+1} - v_{j+1}}$

$$U^{(1)} = \{\underbrace{u_0^{(1)}, \cdots, u_{p-1}^{(1)}}_{p \text{ times}}, \cdots, \underbrace{u_{m-p-1}^{(1)}, \cdots, u_{m-2}^{(1)}}_{p \text{ times}}\} \text{ with } u_i^{(1)} = u_{i+1}, 0 \le i \le m-2$$

$$V^{(1)} = \{\underbrace{v_0^{(1)}, \cdots, v_{q-1}^{(1)}}_{q \text{ times}}, \cdots, \underbrace{v_{n-q-1}^{(1)}, \cdots, v_{n-2}^{(1)}}_{q \text{ times}}\} \text{ with } v_j^{(1)} = v_{j+1}, 0 \le j \le n-2$$

- General case :

$$\frac{\partial^{k+l} S(u,v)}{\partial u^k \partial v^l} = \sum_{i=0}^{n-k} \sum_{j=0}^{m-l} N_i^{p-k}(u) N_j^{q-l}(v) P_{ij}^{(k,l)}$$

$$\text{with} \quad P_{ij}^{(k,l)} = (q-l+1) \frac{P_{i\,j+1}^{(k,l-1)} - P_{ij}^{(k,l-1)}}{v_{j+q+1} - v_{j+l}}$$

$$U^{(k)} = \{\underbrace{u_0^{(k)}, \cdots, u_{p-k}^{(k)}}_{p+1-k \text{ times}}, \cdots, \underbrace{u_{m-p-k}^{(k)}, \cdots, u_{m-2k}^{(k)}}_{p+1-k \text{ times}}\} \text{ with } u_i^{(k)} = u_{i+k}, 0 \le i \le m-2k$$

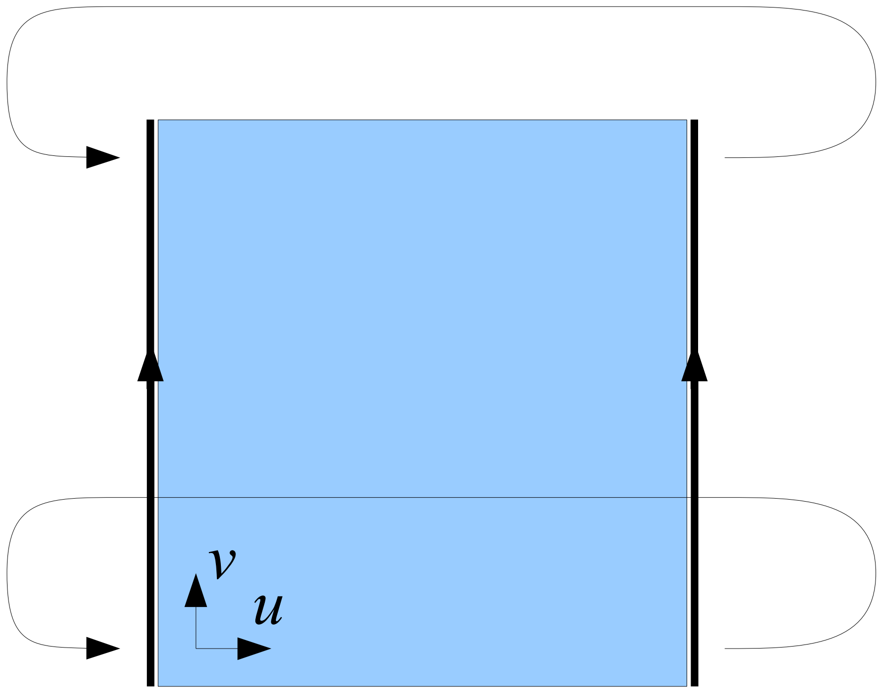
$$V^{(l)} = \{\underbrace{v_0^{(l)}, \cdots, v_{q-l}^{(l)}}_{q+1-l \text{ times}}, \cdots, \underbrace{v_{n-q-l}^{(l)}, \cdots, v_{n-2l}^{(l)}}_{q+1-l \text{ times}}\} \text{ with } v_j^{(l)} = v_{j+l}, 0 \le j \le n-2l$$

- The derivative vector of a B-Spline surface also is a B-Spline surface...

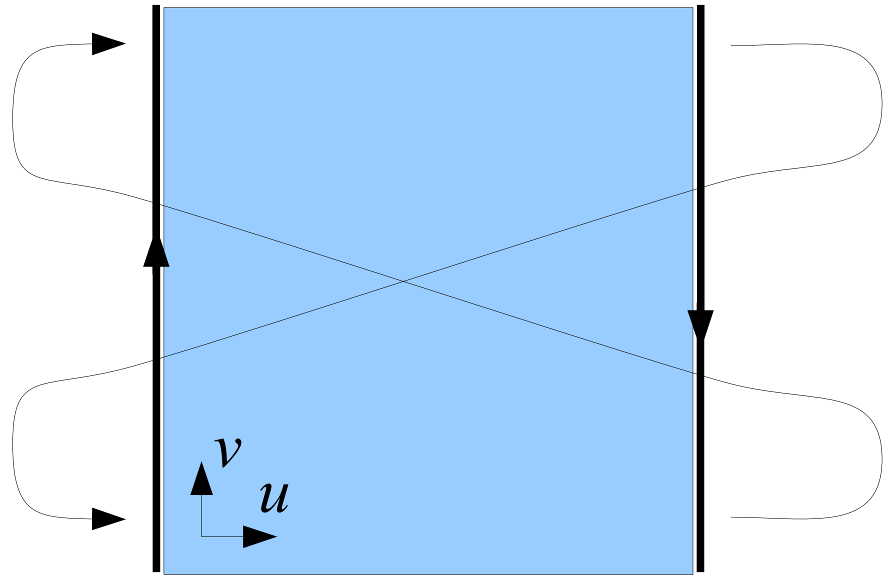
B-Spline surfaces

- Periodic surfaces
  - Like for the curves, possibility to “close” a B-Spline surface by transforming the nodal sequence
    - According to one parameter (u or v)
      Cylindrical surfaces
      - A single periodic nodal sequence
      - Control points on both sides of the seam are doubled
    - According to both parameters (u and v)
      Toroidal surfaces
      - Two periodic nodal sequences
      - Some control points are repeated 4 times !

B-Spline surfaces

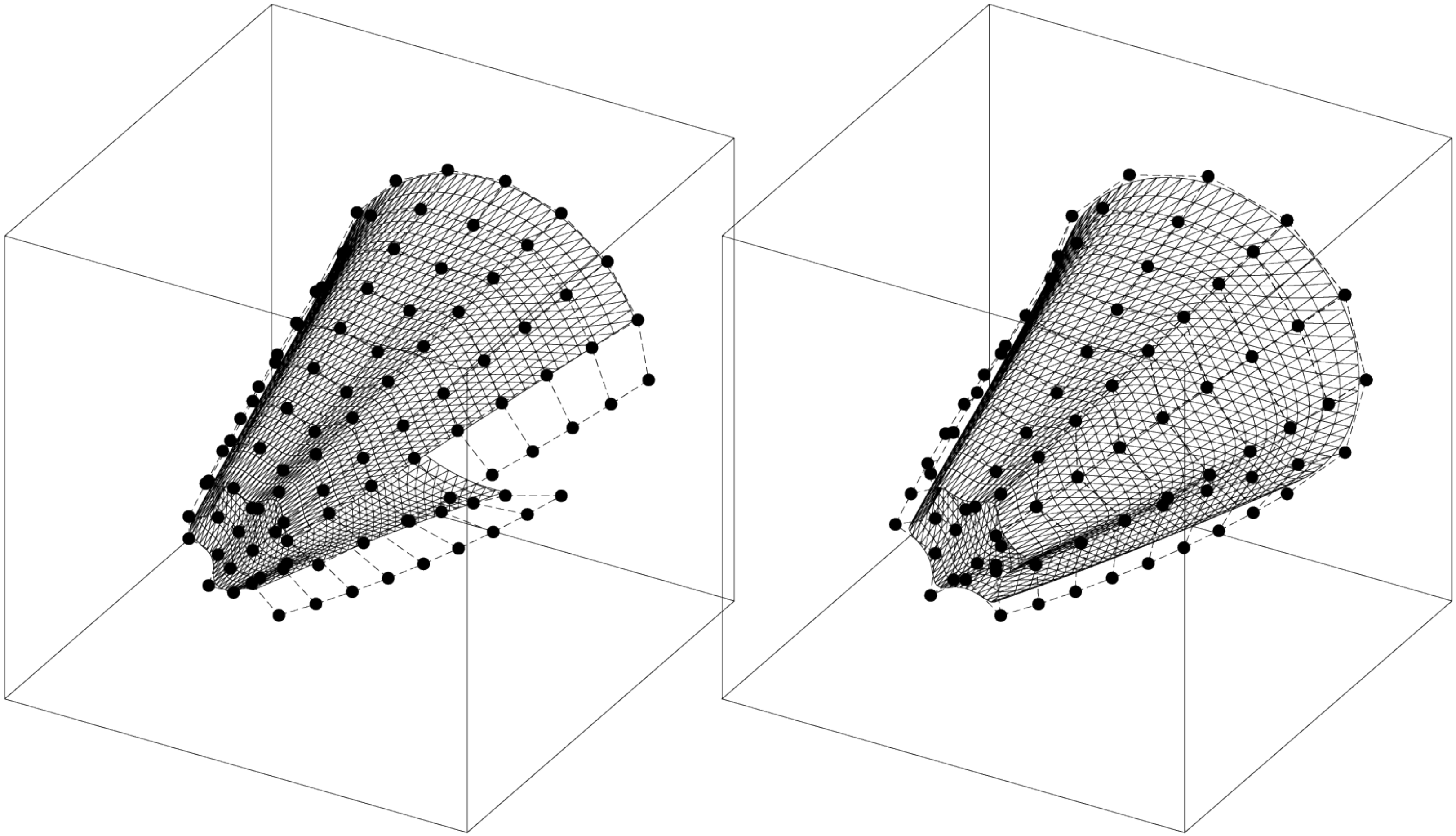


Pipe



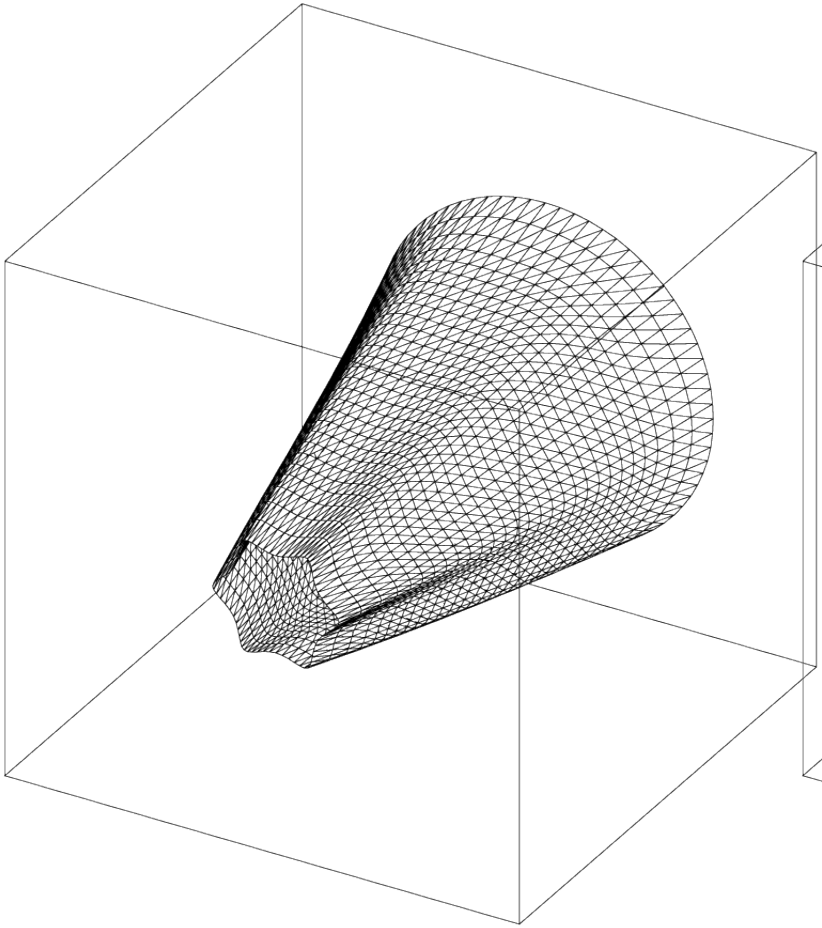
Möbius strip

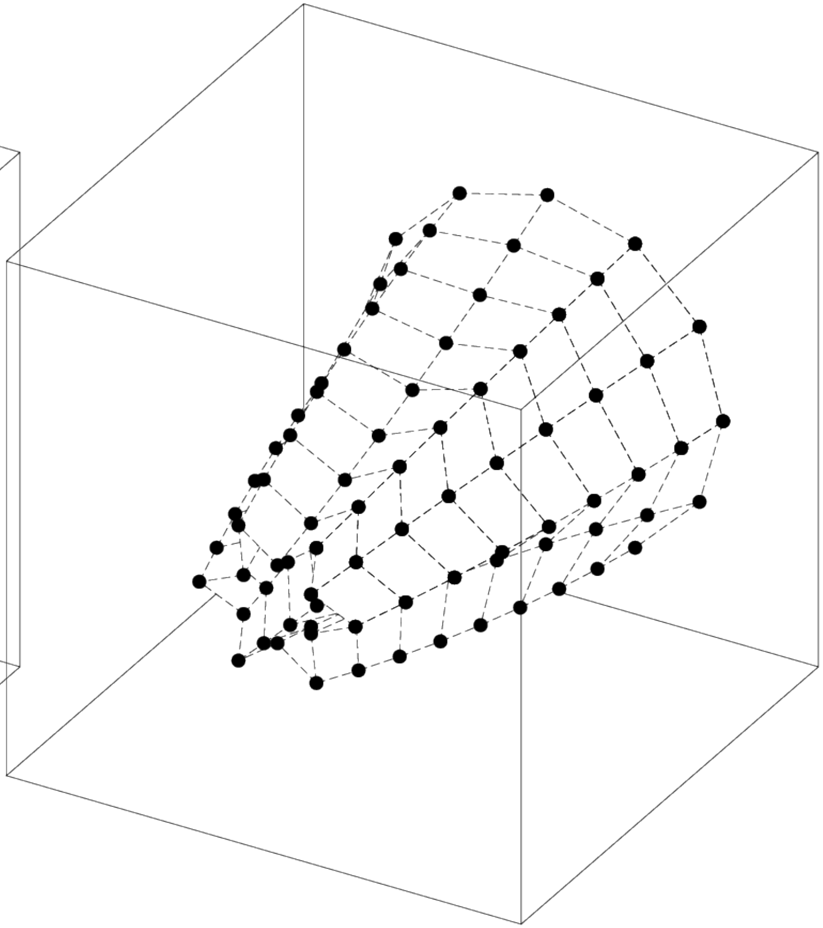
B-Spline surfaces



$$U=\{-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}\quad p=3$$
$$V=\{0,0,0,1,2,3,4,5,6,7,7,7\}\quad p=2$$

B-Spline surfaces

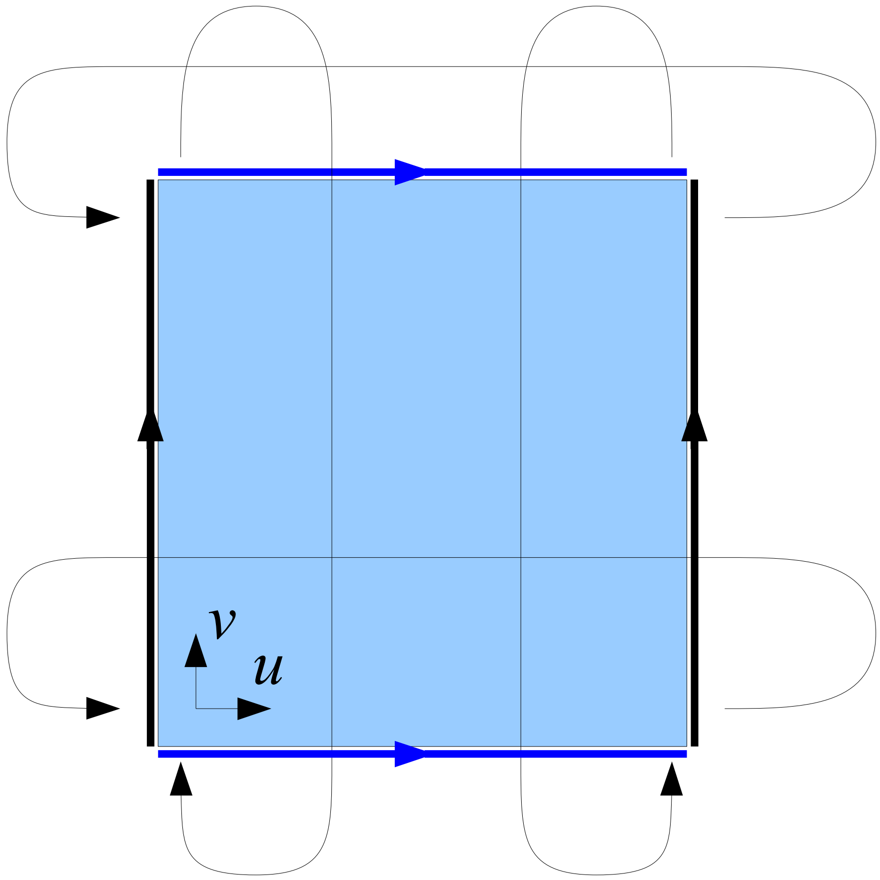




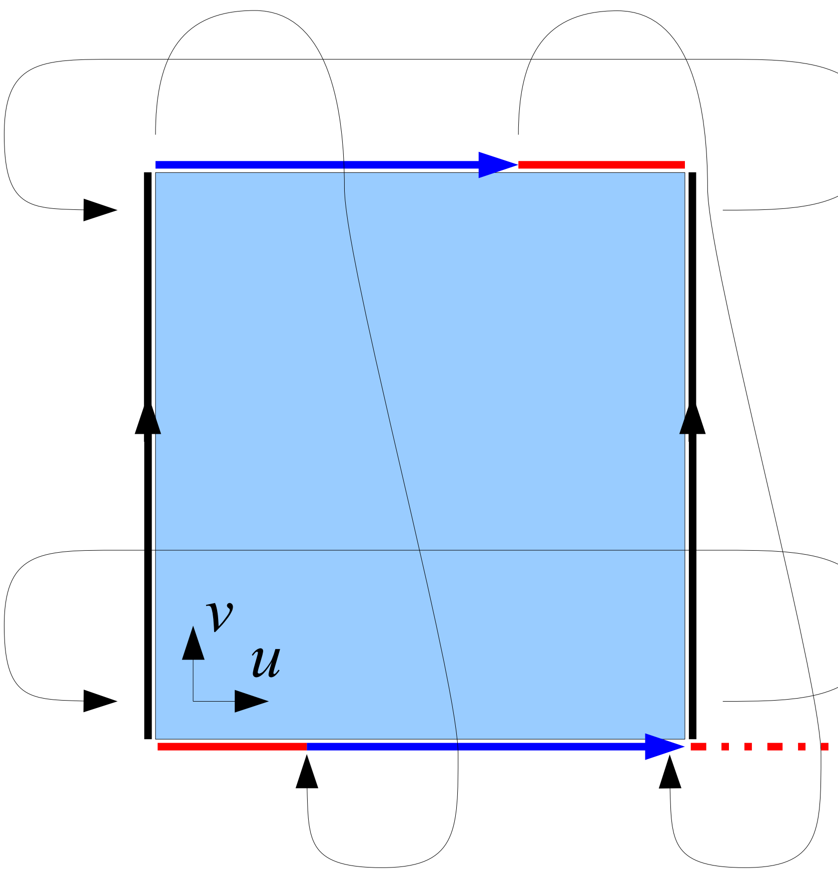
$$U = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \quad p = 3$$

$$V = \{0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7\} \quad p = 2$$

B-Spline surfaces

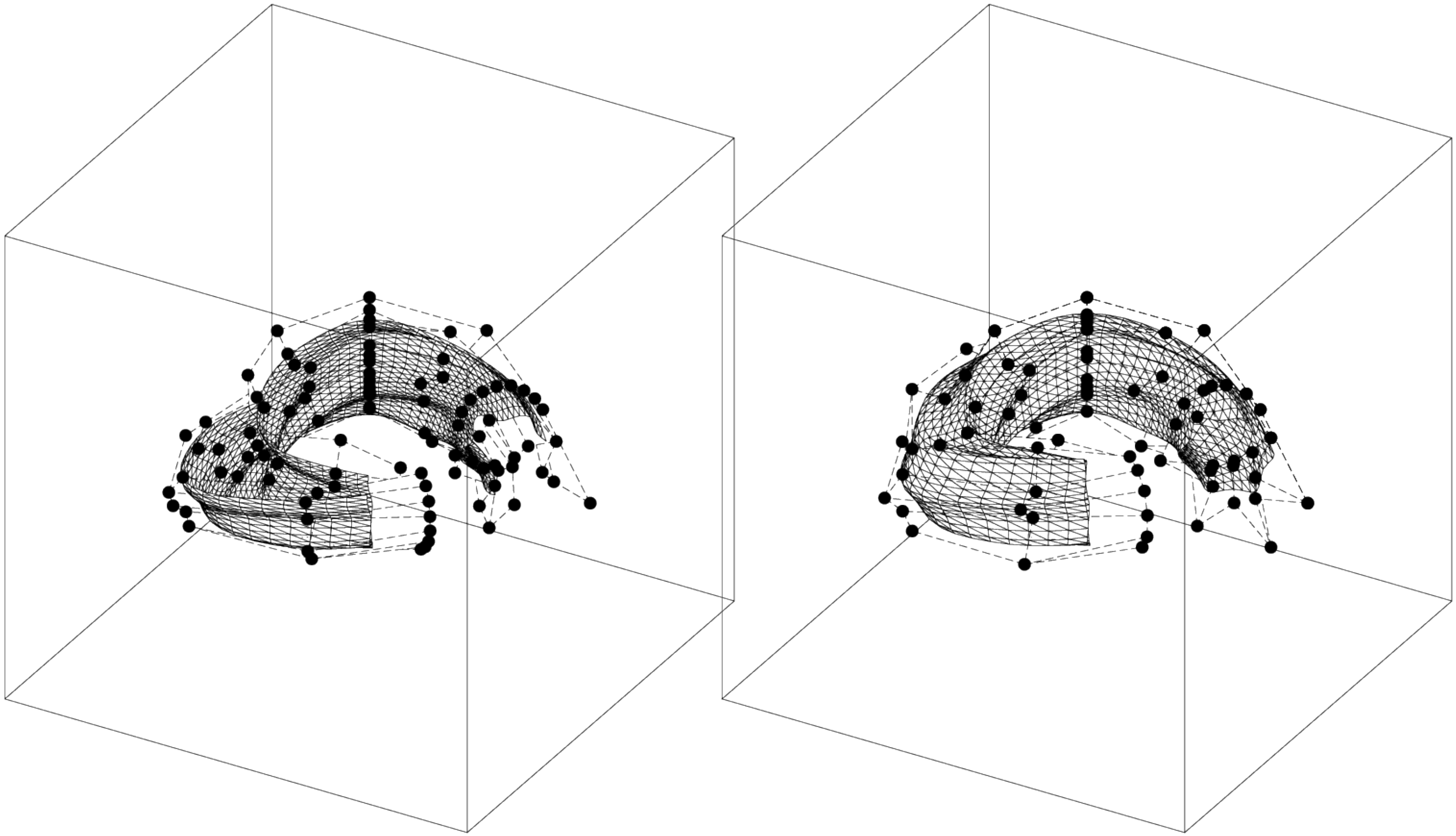


Tore

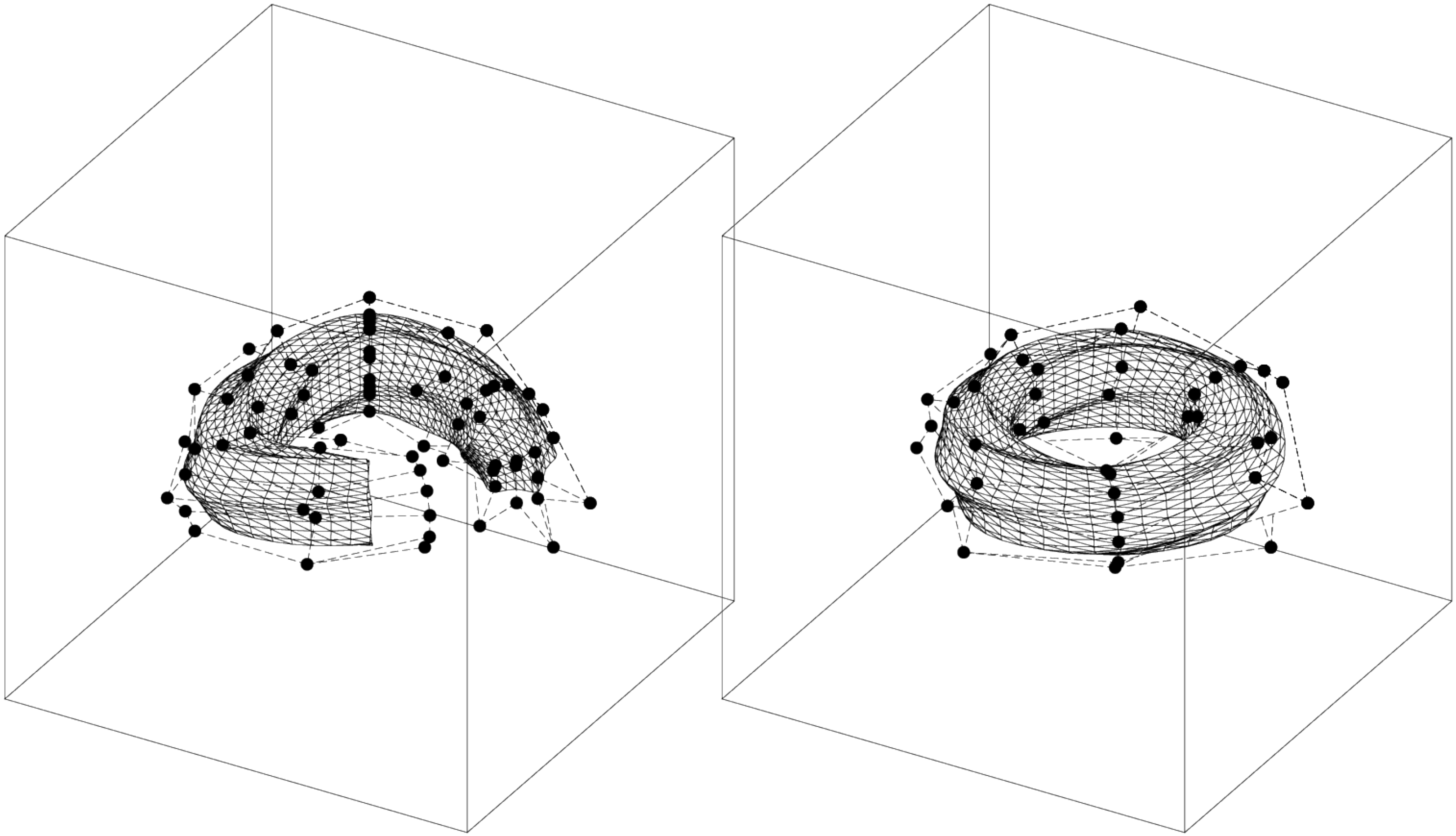


Twisted Tore...

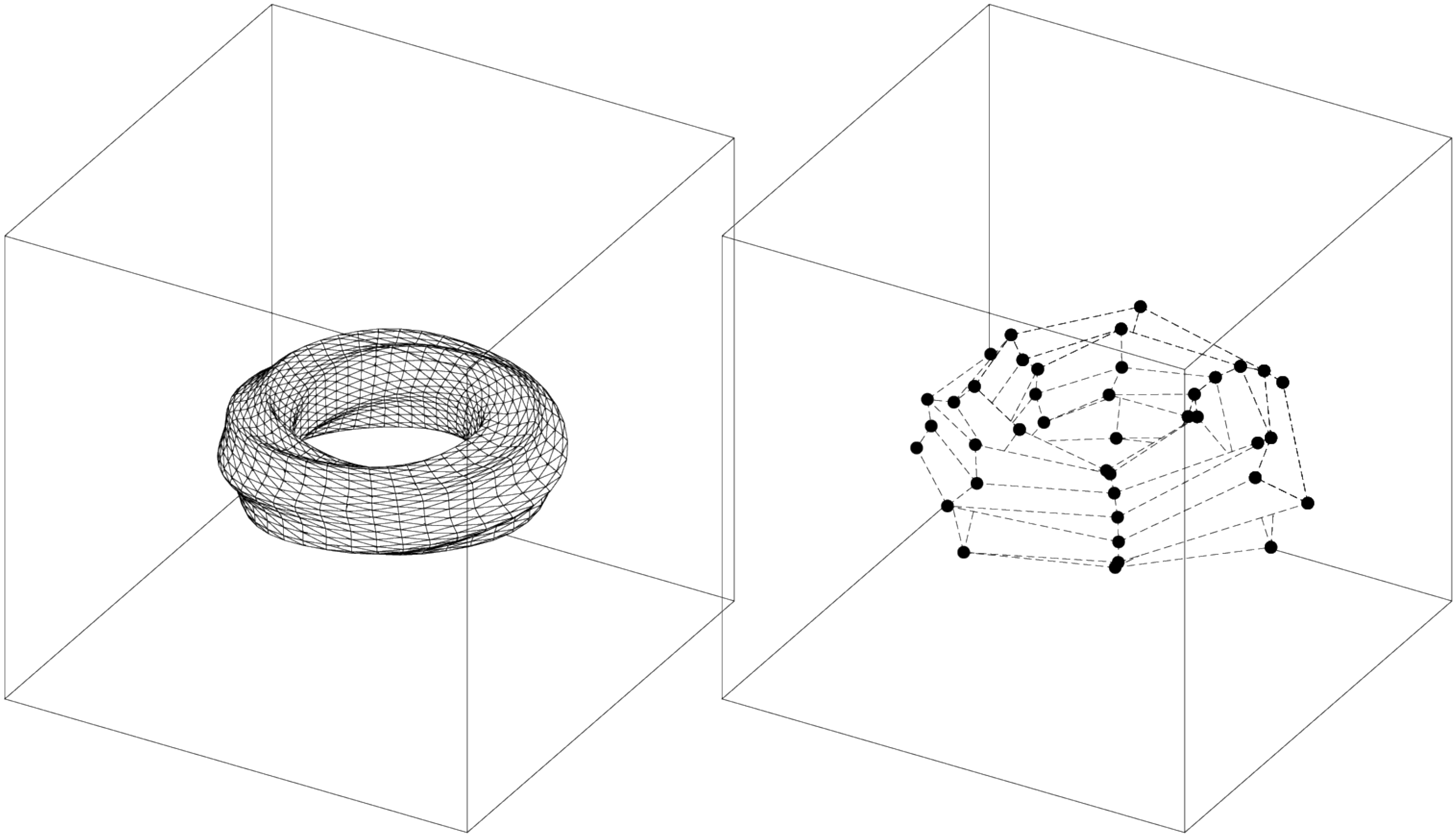
B-Spline surfaces



B-Spline surfaces



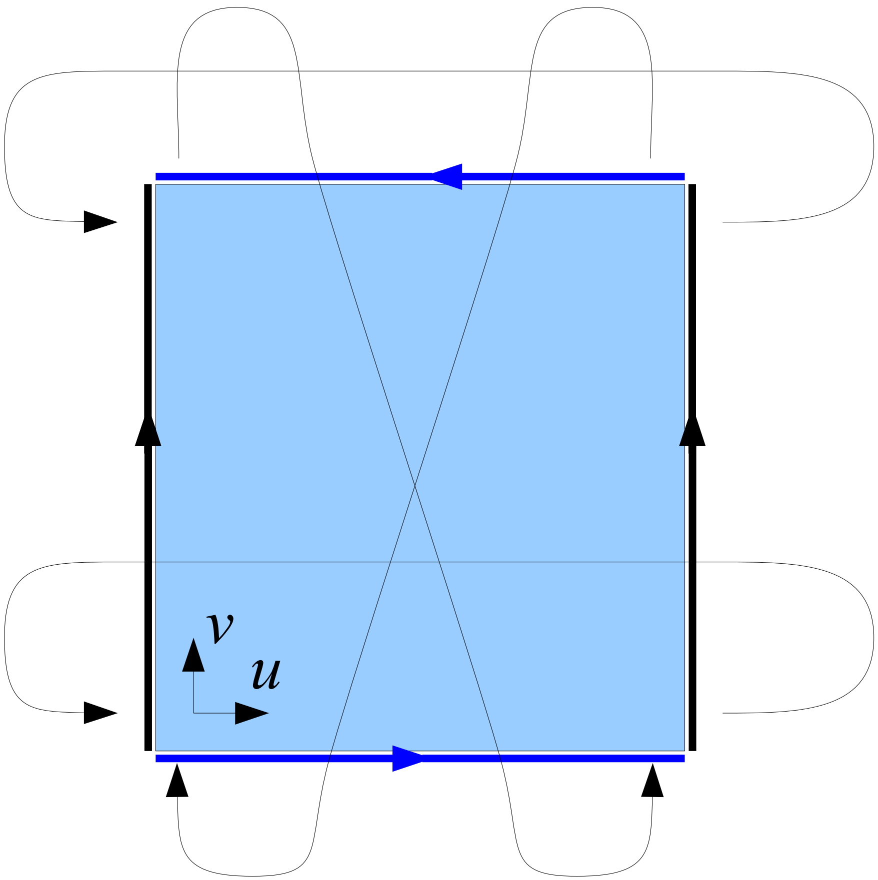
B-Spline surfaces



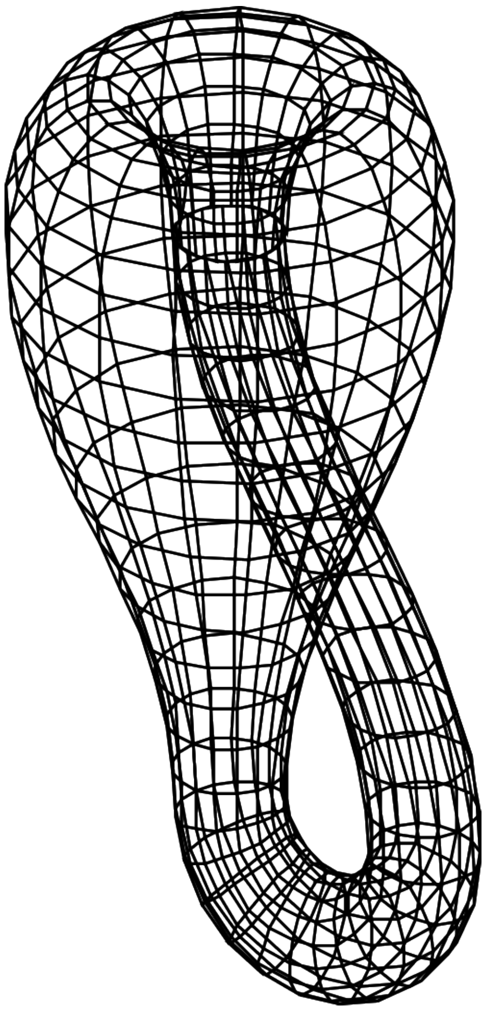
$$U = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \quad p = 3$$

$$V = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \quad p = 2$$

B-Spline surfaces



Klein's bottle



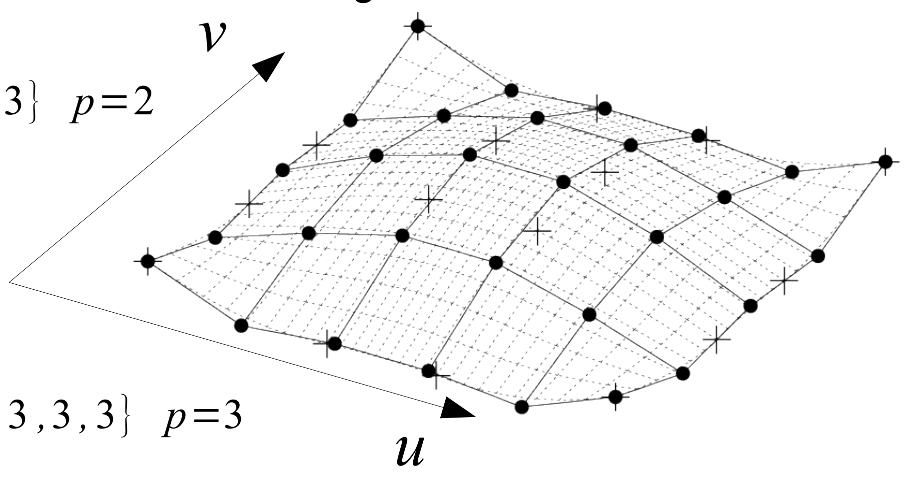
B-Spline surfaces

- Some manipulations
  - Insertion of nodes
    - Extraction of iso-parametrics
    - Calculation of the position of a point on the surface
    - Subdivision of the surface
    - Transformation into Bézier patches

B-Spline surfaces

- Insertion of nodes
  - We insert nodes In a nodal sequence (U ou V)
  - The new nodal sequence replace the old one
  - The control points are modified
    - If U is modified, every series of control points corresponding to $v\!=\!cst$ is independently modified
    - If V is modified, every series of control points corresponding to $u\!=\!cst$ is independently modified
    - We use Boehm's algorithm as for curves

$V=\{0,0,0,1,2,3,3,3\} \quad p=2$

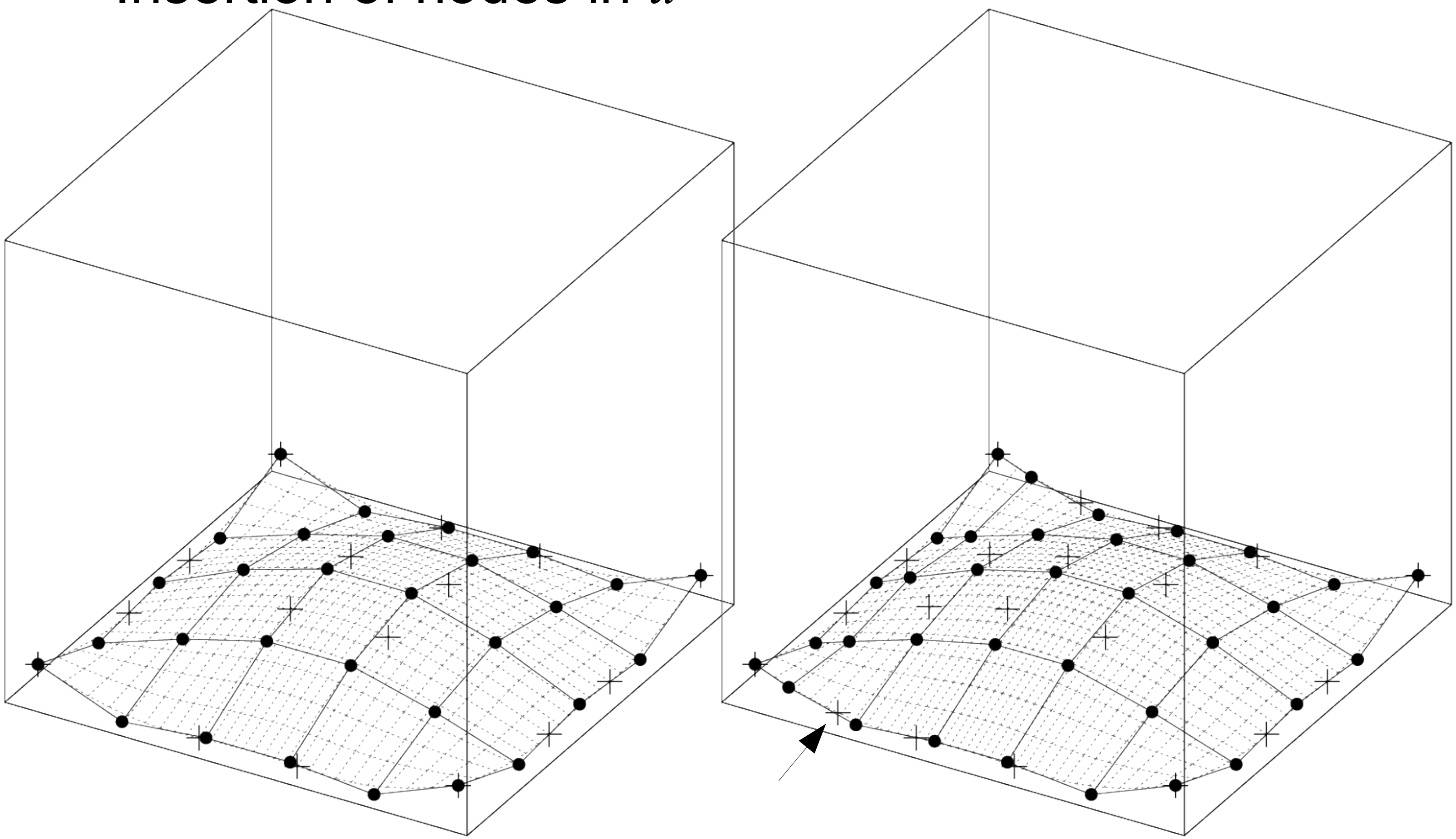


$U=\{0,0,0,0,1,2,3,3,3,3\} \quad p=3$

Computer Aided Design

B-Spline surfaces

- Insertion of nodes in u



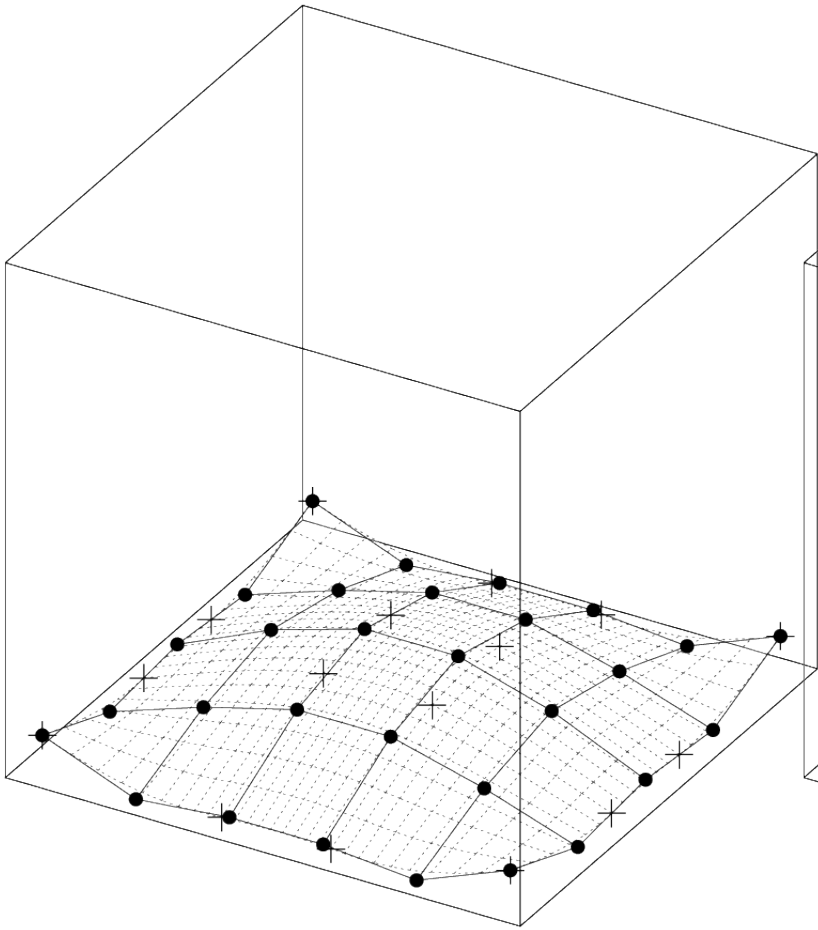
$U=\{0,0,0,0,1,2,3,3,3,3\}$ $p=3$
$V=\{0,0,0,1,2,3,3,3\}$ $q=2$

$U=\{0,0,0,0,0.4,1,2,3,3,3,3\}$ $p=3$
$V=\{0,0,0,1,2,3,3,3\}$ $q=2$

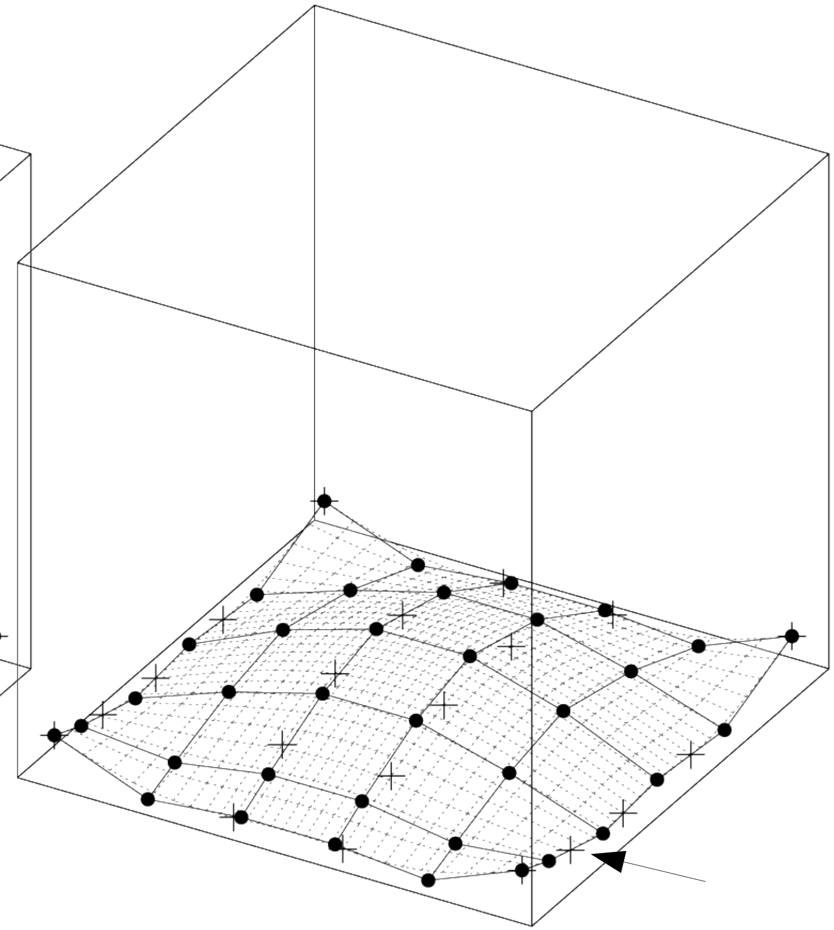
Computer Aided Design

B-Spline surfaces

- Insertion of nodes in v



$U=\{0,0,0,0,1,2,3,3,3,3\}\quad p=3$
$V=\{0,0,0,1,2,3,3,3\}\quad q=2$



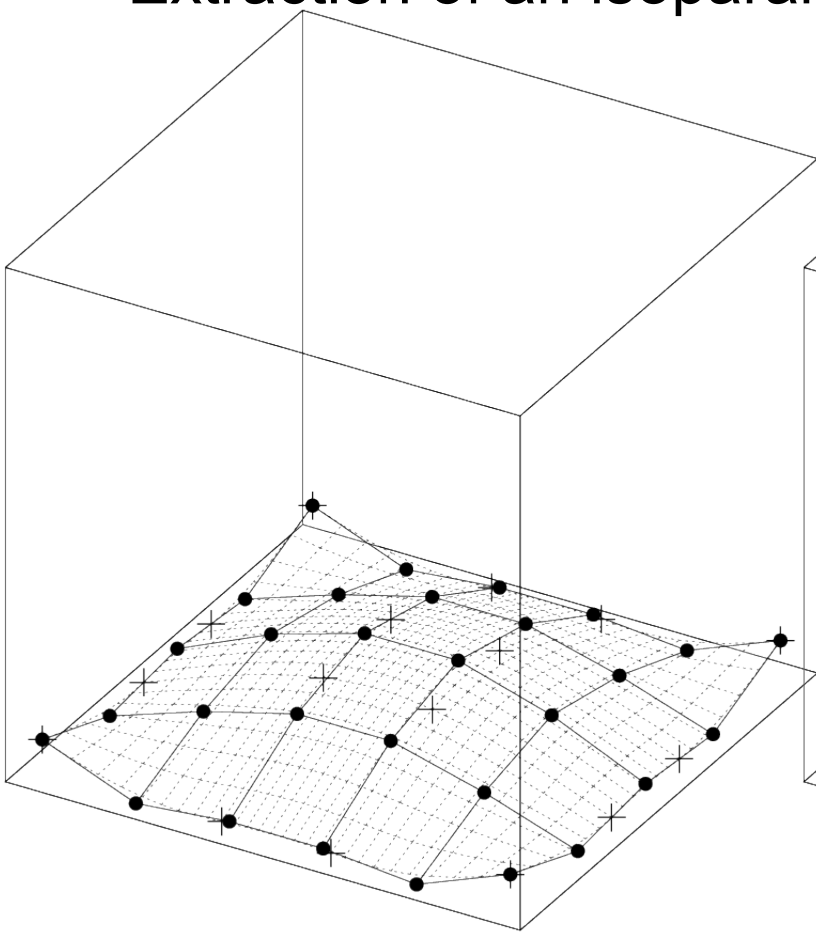
$U=\{0,0,0,0,1,2,3,3,3,3\}\quad p=3$
$V=\{0,0,0,0.4,1,2,3,3,3\}\quad q=2$

B-Spline surfaces

- Extraction of iso-parametrics using node insertion
  - We must saturate one node in $u=u_{iso}$ (resp. in $v=v_{iso}$).
  - The new control points obtained by Boehm's algorithm do form the control polygon of the iso-parametric curve.
  - The nodal sequence of this curve is V (resp. U).

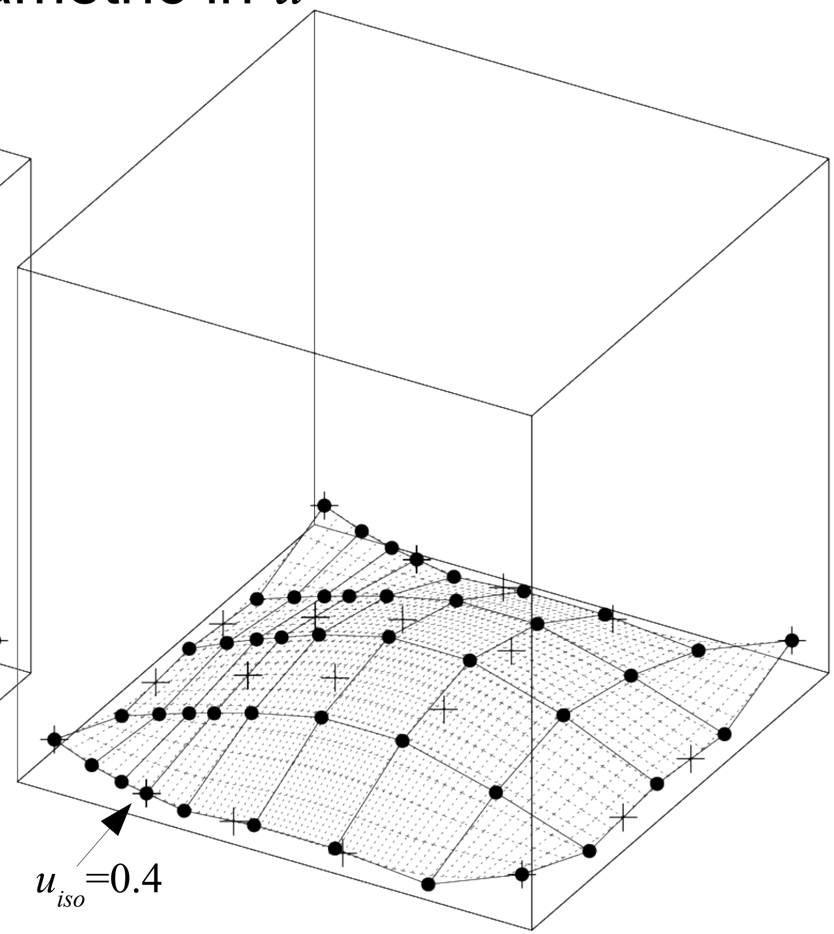
B-Spline surfaces

- Extraction of an isoparametric in u



$U=\{0,0,0,0,1,2,3,3,3,3\}\quad p=3$

$V=\{0,0,0,1,2,3,3,3\}\quad q=2$



$U=\{0,0,0,0,0.4,0.4,0.4,1,2,3,3,3,3\}\quad p=3$

$V=\{0,0,0,1,2,3,3,3\}\quad q=2$

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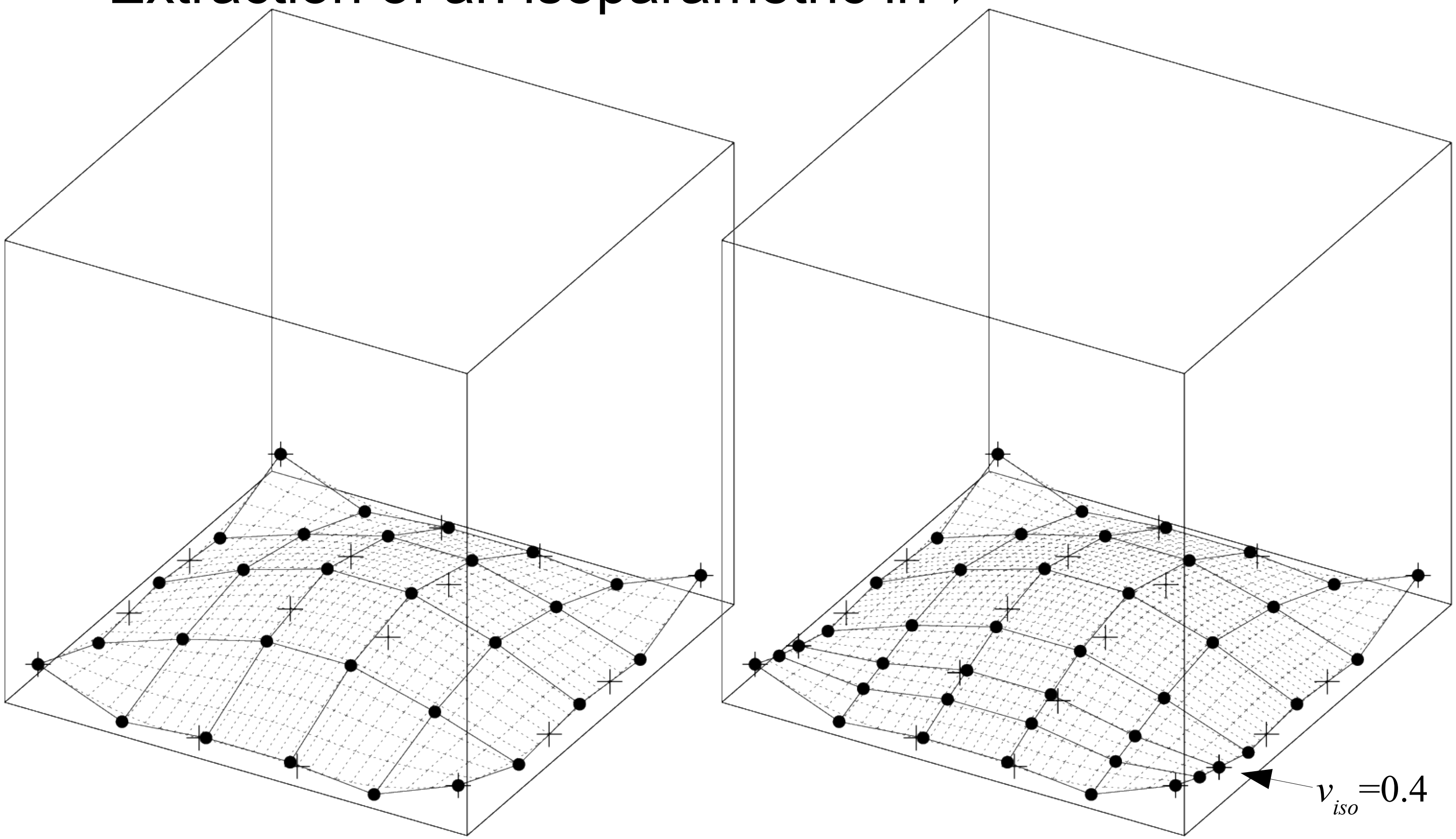
B-Spline surfaces

- Extraction of an isoparametric in v

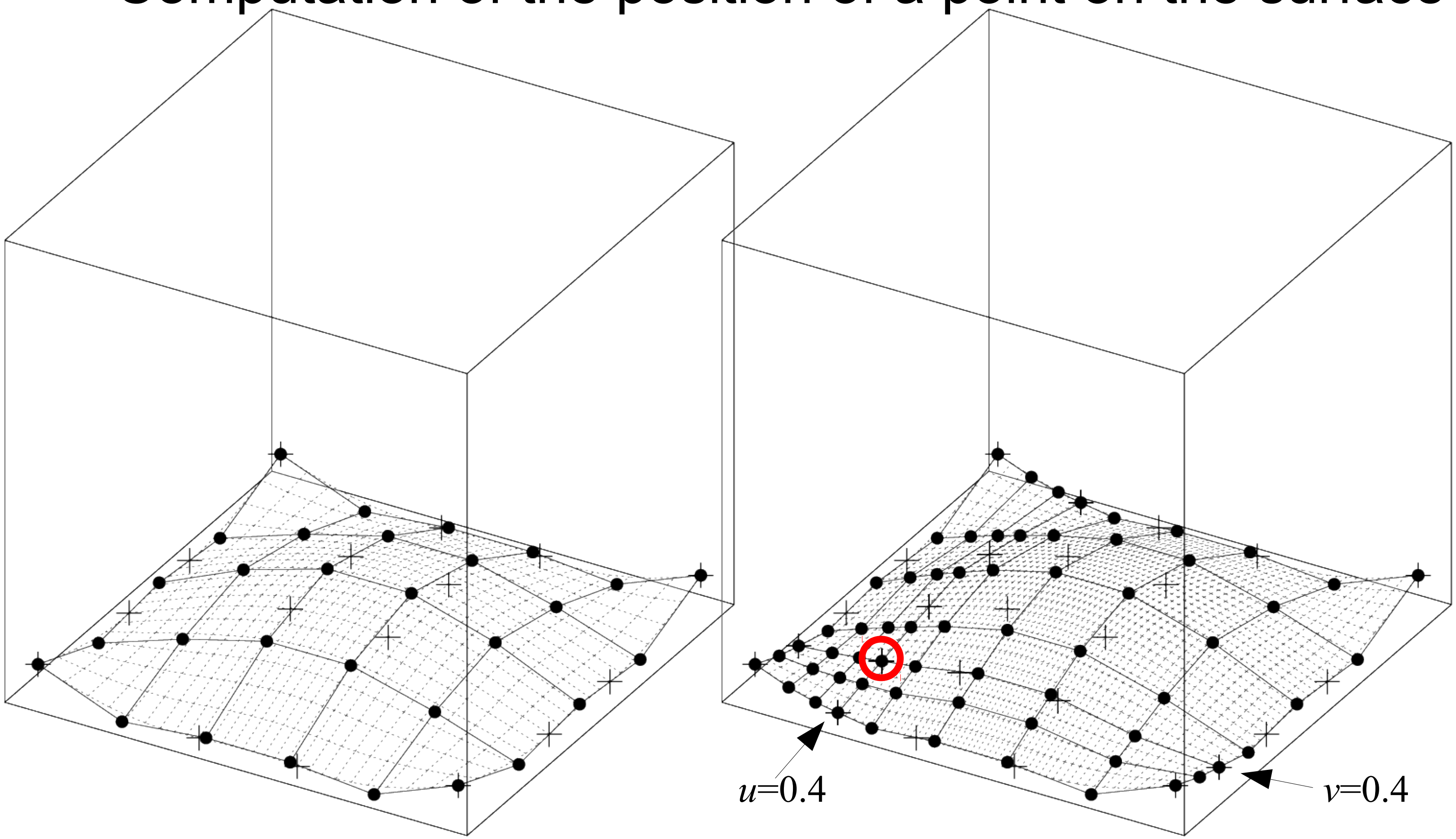


$U=\{0,0,0,0,1,2,3,3,3,3\}$ $p=3$
$V=\{0,0,0,1,2,3,3,3\}$ $q=2$

$U=\{0,0,0,0,1,2,3,3,3,3\}$ $p=3$
$V=\{0,0,0,0.4,0.4,1,2,3,3,3\}$ $q=2$

B-Spline surfaces

- Computation of the position of a point on the surface

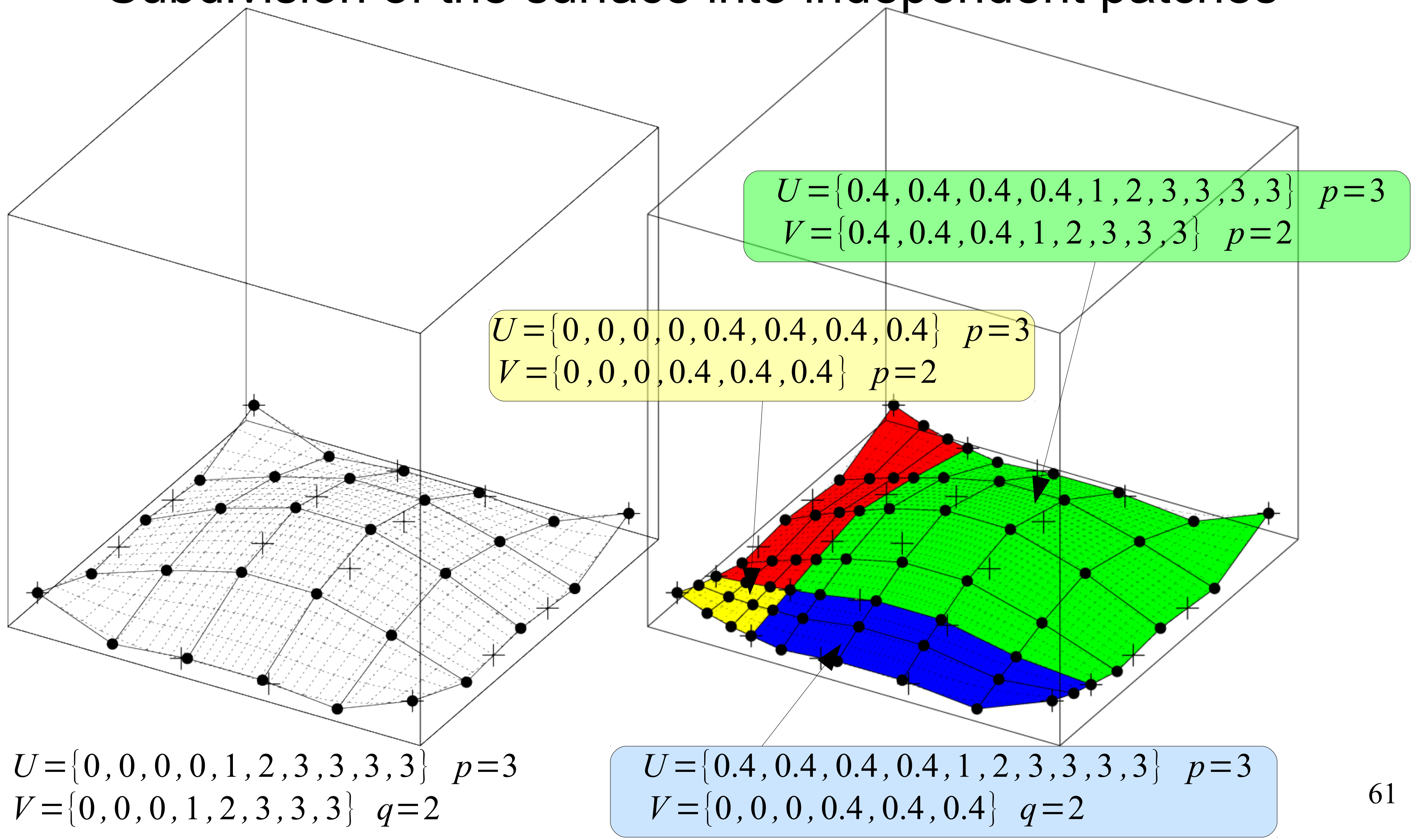


$U=\{0,0,0,0,1,2,3,3,3,3\}\quad p=3$
$V=\{0,0,0,1,2,3,3,3\}\quad q=2$

$U=\{0,0,0,0,0.4,0.4,0.4,1,2,3,3,3,3\}\quad p=3$
$V=\{0,0,0,0.4,0.4,1,2,3,3,3\}\quad q=2$

B-Spline surfaces

- Subdivision of the surface into independent patches



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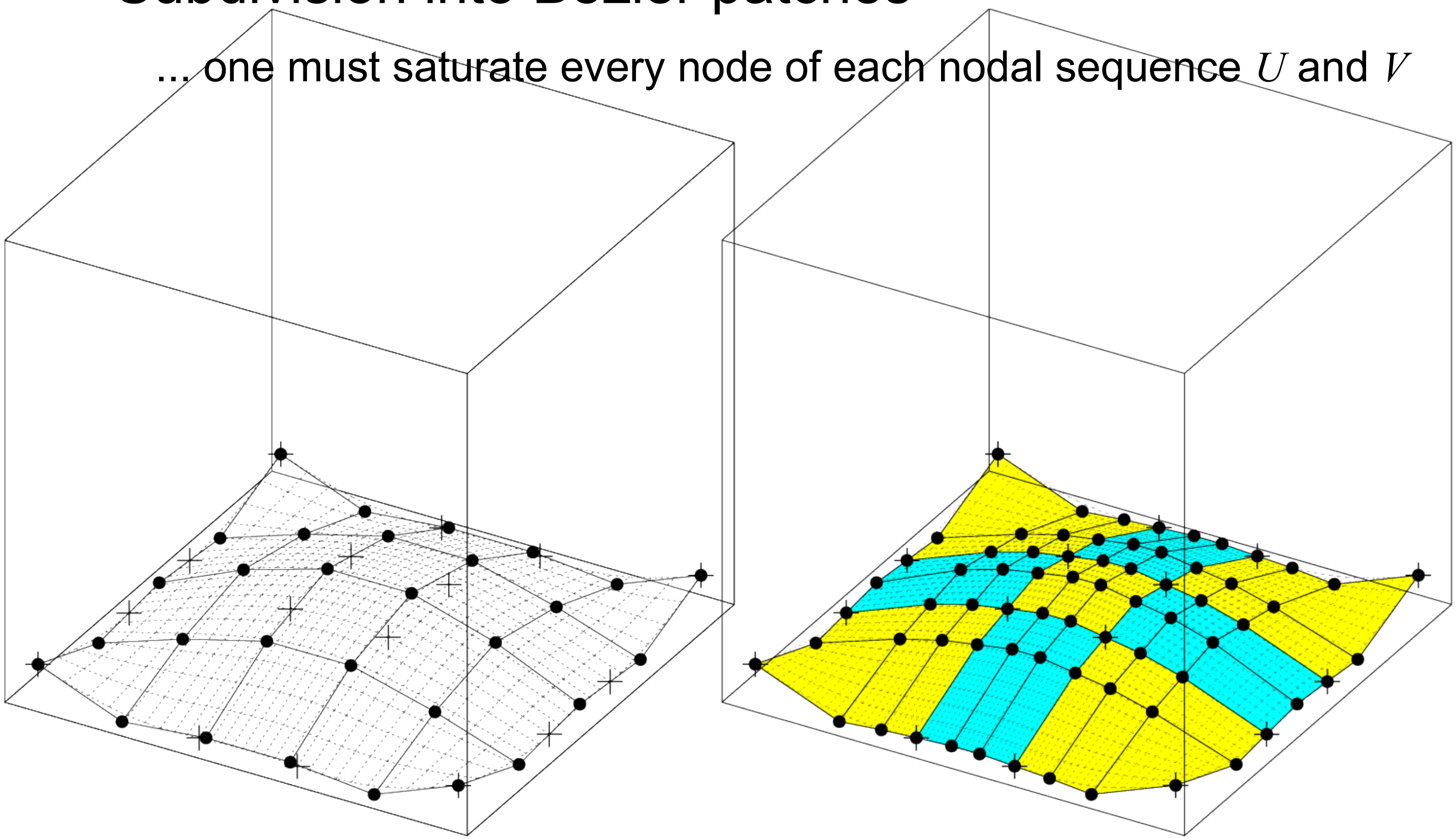
B-Spline surfaces

- Subdivision into Bézier patches

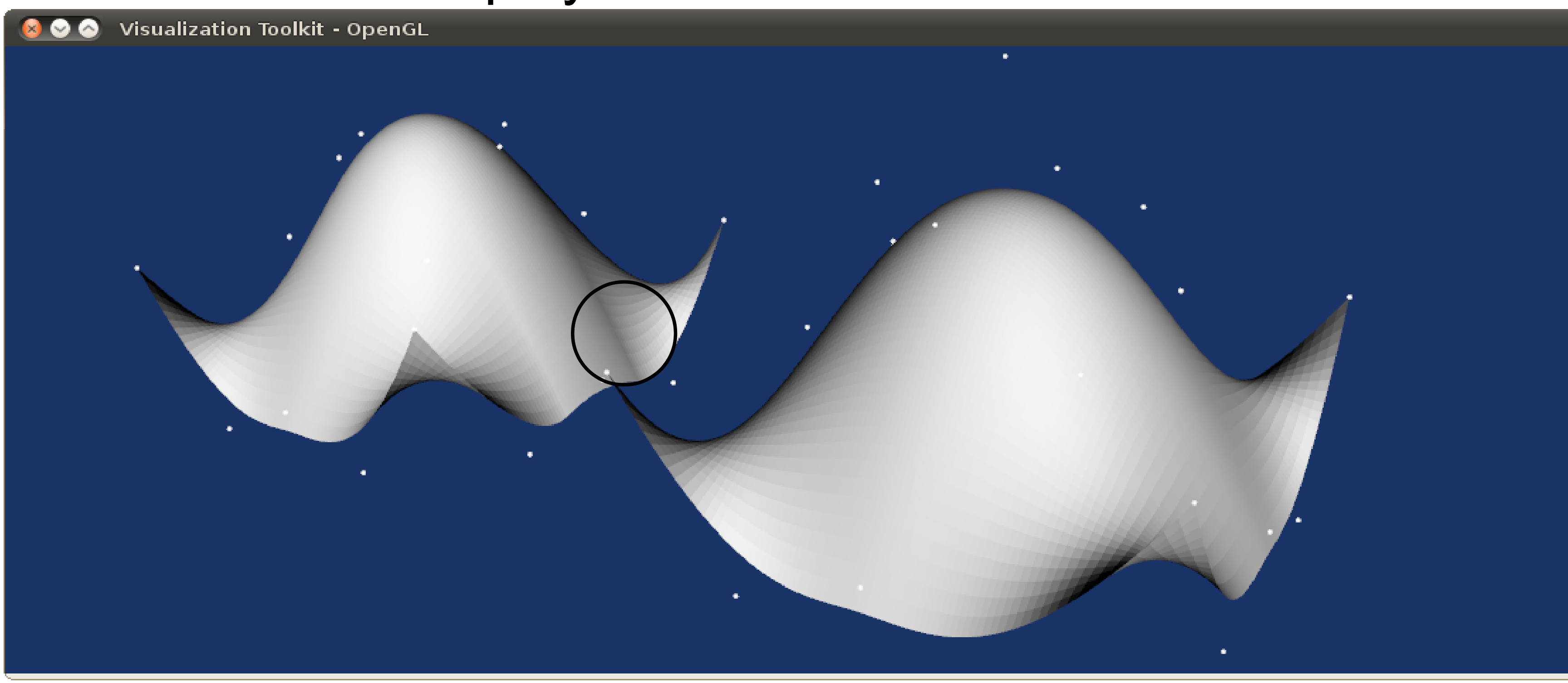
  ... one must saturate every node of each nodal sequence U and V



$U=\{0,0,0,0,1,2,3,3,3,3\}$ $p=3$
$V=\{0,0,0,1,2,3,3,3\}$ $q=2$

$U=\{0,0,0,0,1,1,1,2,2,2,3,3,3,3\}$ $p=3$
$V=\{0,0,0,1,1,2,2,3,3,3\}$ $q=2$

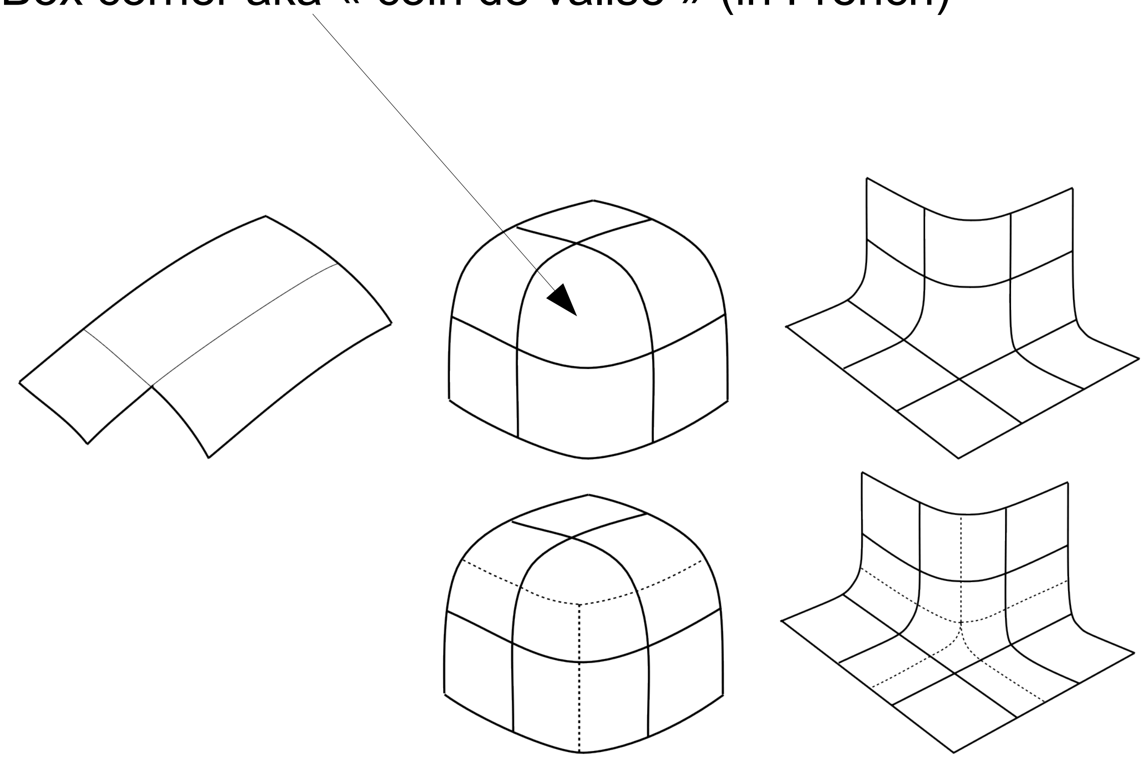
- Continuity requirements for surfaces
  - C^1 vs C^2 – becomes visible when light interaction comes into play



Bézier triangle

Bézier triangle

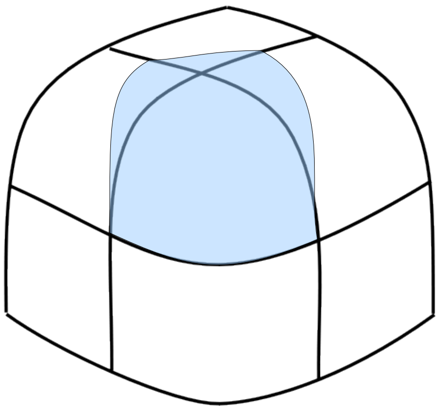
- Need for specific topology
  - Box corner aka « coin de valise » (in French)

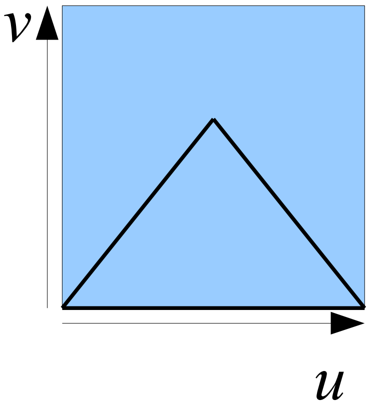


S. Hahmann

Bézier triangle

- There are several techniques to model the corner
  - It's more difficult than one thinks ...
  - On may take a regular patch with 4 sides and « limit » it by a triangle in the parametric space
    - Problem : The surface in question is not built with the control points of the other surfaces, so any continuity is difficult to enforce (minimization of a non linear functional)

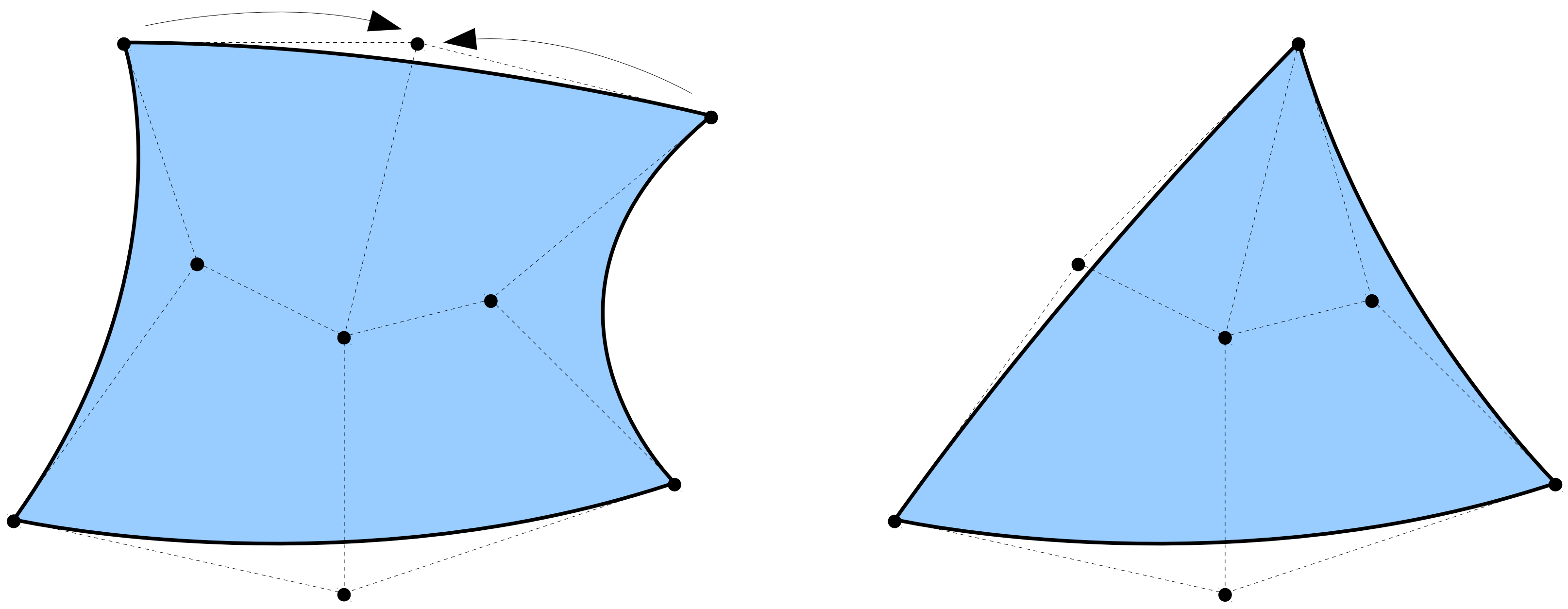




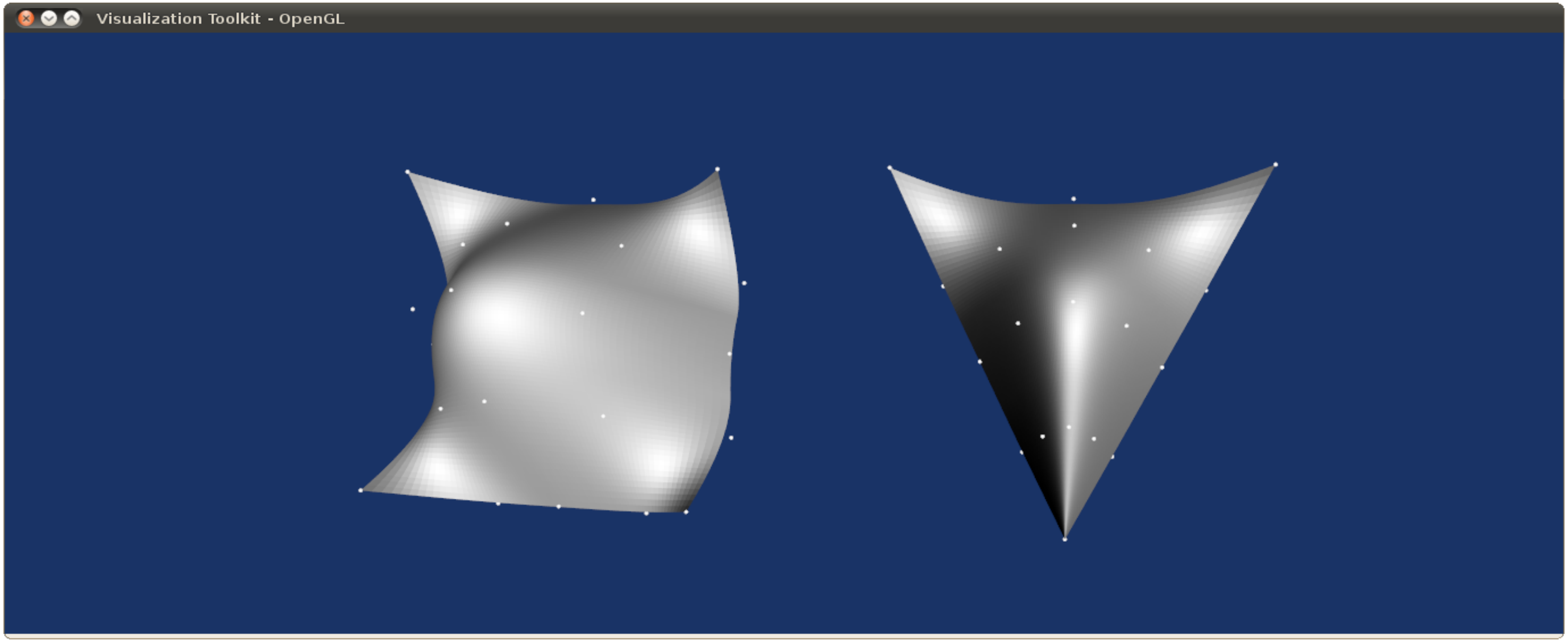
Computer Aided Design

Bézier triangle

- Degenerated quadrangular patch
  - Normals and derivatives are undefined at the singular point



Bézier triangle



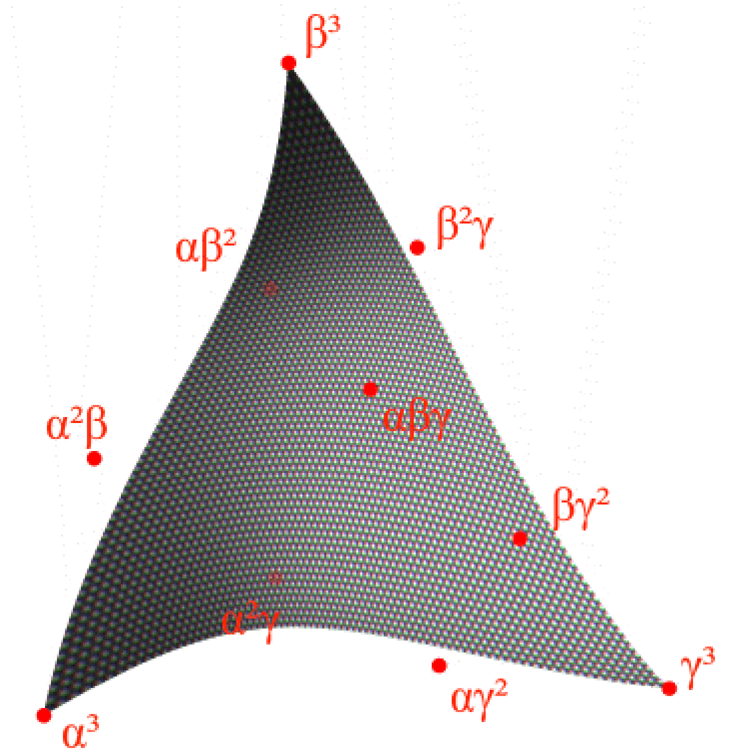
Bézier triangle

- Triangular B-splines
  - 1992 : works of Dahmen, Micchelli et Seidel

    W. Dahmen, C.A. Micchelli and H.P. Seidel, Blossoming begets B-Splines built better by B-patches, *Mathematics of Computation*, 59 (199), pp. 97-115, 1992
  - Extension of the definition of B-Splines on triangular surfaces of any topology
    - Network of control points
    - « Mesh » of non structured topology instead of a structured network as for B-splines surfaces
  - Complex and not usually not implemented in current CAD software, therefore not a “standard” tool.

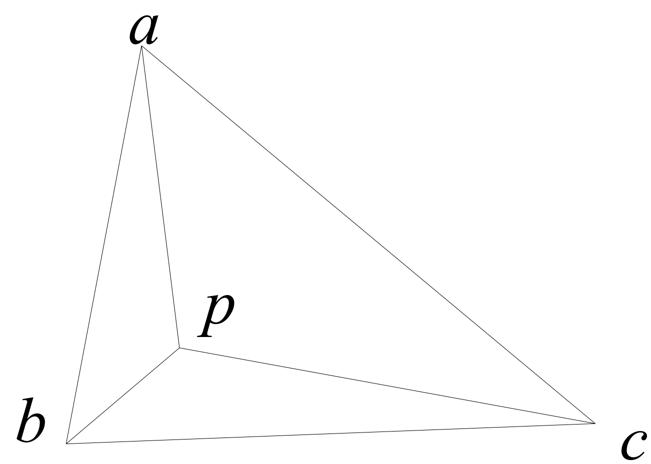
Bézier triangle

- Triangular Bézier Surfaces
  - Example : surface of order 3



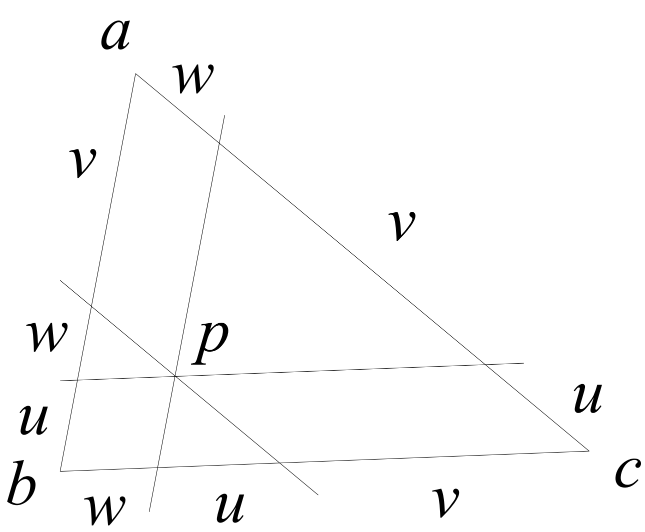
Bézier triangle

- Barycentric coordinates
  - $p = u \cdot a + v \cdot b + w \cdot c$
  - $u + v + w = 1$
  - Affine invariance
  - $0 \leq u, v, w \leq 1 \Leftrightarrow p$ is in the triangle
  - $u = \frac{\text{area}(p,b,c)}{\text{area}(a,b,c)} \quad v = \frac{\text{area}(p,c,a)}{\text{area}(a,b,c)} \quad w = \frac{\text{area}(p,a,b)}{\text{area}(a,b,c)}$



Bézier triangle

- Barycentric coordinates



Bézier triangle

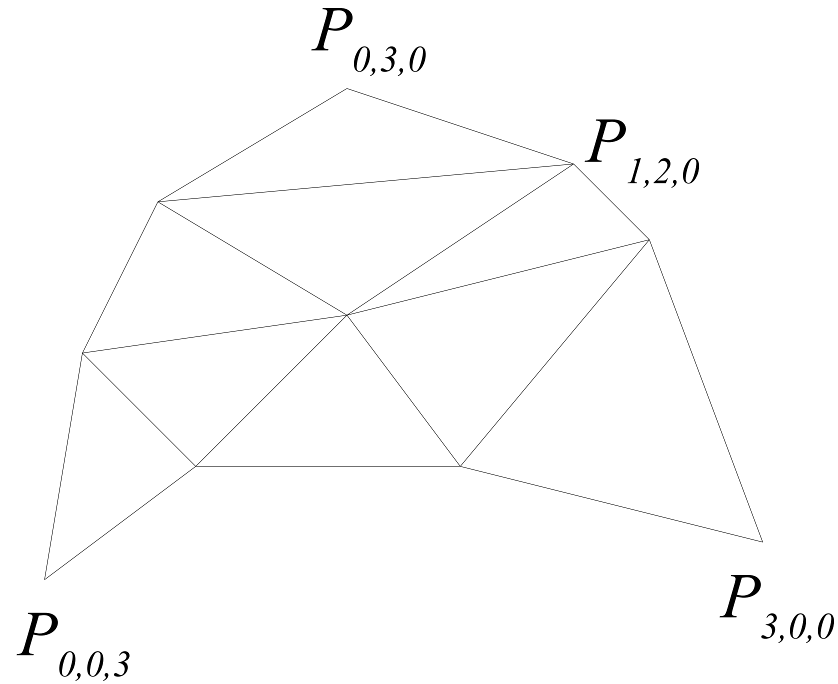
- Decomposition of the Bézier triangle
  - Defined by the control points $P_{i,j,k}$
  - Degree d : $i+j+k=d$
  - Example with $d=3$

$P_{0,3,0}$

$P_{0,2,1}$ $P_{1,2,0}$

$P_{0,1,2}$ $P_{1,1,1}$ $P_{2,1,0}$

$P_{0,0,3}$ $P_{1,0,2}$ $P_{2,0,1}$ $P_{3,0,0}$



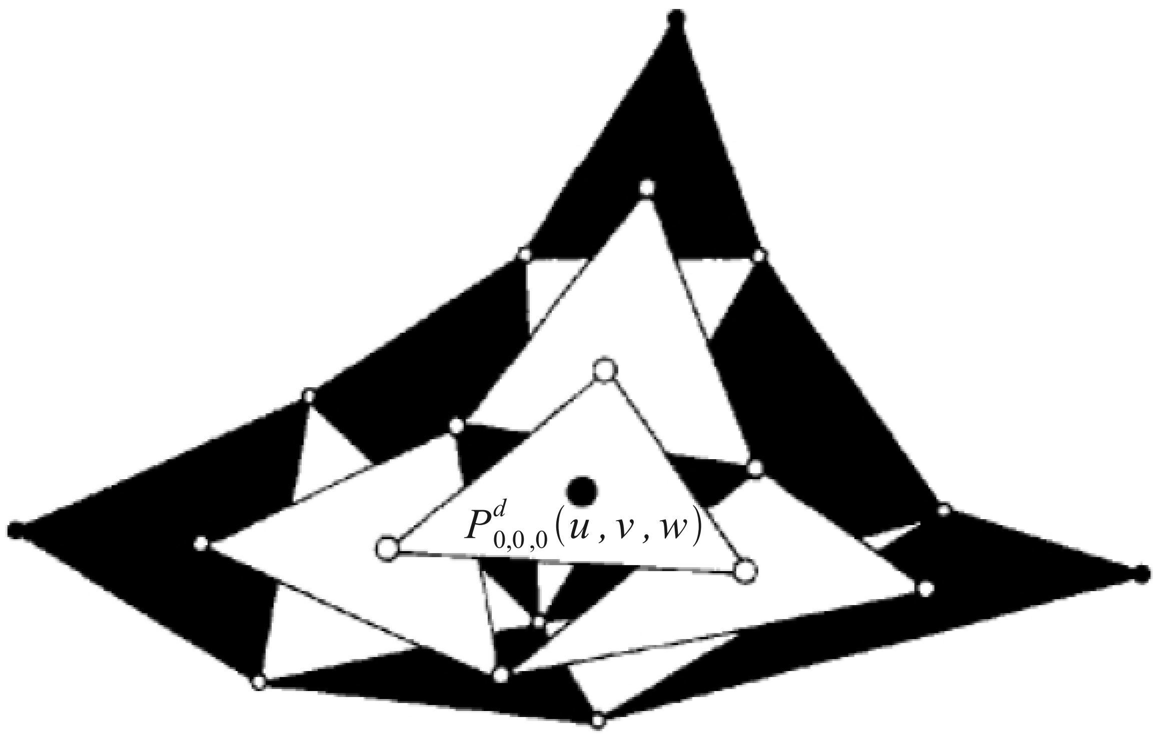
  - Overall, $\frac{(d+2)(d+1)}{2}$ control points.

- De Casteljau's algorithm on the Bézier triangle
  - The $P_{i,j,k}$ are given
  - We want to compute $P(u,v,w)$ with $u+v+w=1$
  - We follow the following algorithm :
    initialize $P^0_{i,j,k}(u,v,w)=P_{i,j,k}$

    for r from 1 to d and for every triplet (i,j,k) s.t. $i+j+k=d\text{-}r$
    $$P^r_{i,j,k}(u,v,w)=u\,P^{r-1}_{i+1,j,k}(\cdots)+v\,P^{r-1}_{i,j+1,k}(\cdots)+w\,P^{r-1}_{i,j,k+1}(\cdots)$$
    The point on the surface is the last point :
    $$P(u,v,w)=P^d_{0,0,0}(u,v,w)$$

Bézier triangle

- De Casteljau's algorithm on the Bézier triangle



Bézier triangle

- Characteristics of the Bézier triangle
  - Affine invariance
  - Contained in the convex hull of the control points
  - Interpolation of extremal vertices
  - Edges of the Bézier triangle are in fact Bézier curves
  - Algebraic form : another form of Bernstein polynomials

$$P(u,v,w)=\sum_{i+j+k=d} B^d_{i,j,k}(u,v,w)\,P_{i,j,k}$$

with (recurrence) $B^d_{i,j,k}(u,v,w)=\dfrac{d!}{i!\,j!\,k!}u^i v^j w^k$

$$B^d_{i,j,k}(u,v,w)=uB^{d-1}_{i-1,j,k}(\cdots)+vB^{d-1}_{i,j-1,k}(\cdots)+wB^{d-1}_{i,j,k-1}(\cdots)$$

$$B^0_{0,0,0}(u,v,w)=1$$

Bézier triangle

- Characteristics (following)
  - On the contrary to tensor product surfaces, the Bézier triangle is **variation diminishing**.

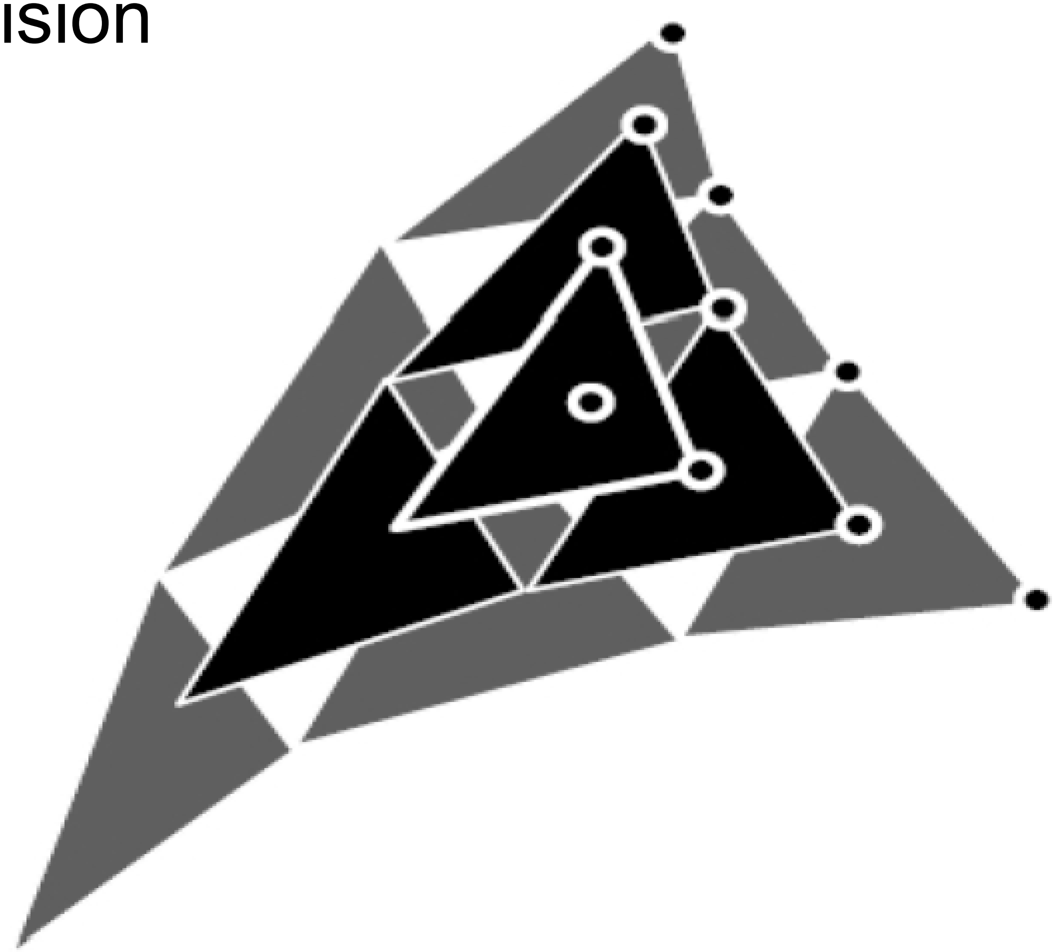
- Degree elevation
  - The $P_{i,j,k}$ are given, we search the $P'_{i,j,k}$ corresponding to the same surface of degree $d+1$
  - Forrest's relations for the Bézier triangle

$$P'_{i,j,k}=\frac{1}{d+1}\left(i\,P_{i-1,j,k}+j\,P_{i,j-1,k}+k\,P_{i,j,k-1}\right)$$

$$P(u,v,w)=\sum_{i+j+k=d+1} B^{d+1}_{i,j,k}(u,v,w)\,P'_{i,j,k}$$
$$=\sum_{i+j+k=d} B^{d}_{i,j,k}(u,v,w)\,P_{i,j,k}$$

Bézier triangle

- Subdivision



Bézier triangle

- Derivatives
  - For tensor product surfaces, partial derivatives are computed along u=const or v=const
  - Here, we express directional derivatives for u=const; v=const or w=const. - these are not partial derivatives !

$$D_u(P(u,v,w)) = \lim_{t \to 0} \frac{P(u, v+tdv, w+tdw) - P(u,v,w)}{t}$$

Bézier triangle

- Case of a surfaces of degree 3

$$
\begin{matrix}
 & & & P_{0,3,0} & & & \\
 & & P_{0,2,1} & & P_{1,2,0} & & \\
 & P_{0,1,2} & & P_{1,1,1} & & P_{2,1,0} & \\
P_{0,0,3} & & P_{1,0,2} & & P_{2,0,1} & & P_{3,0,0}
\end{matrix}
$$

$$P(0,1,0)=P_{0,3,0}$$

$$D_u(P(u,v,w))(0,1,0)=3(P_{0,2,1}-P_{0,3,0})$$

$$D_v(P(u,v,w))(0,1,0)=3(P_{1,2,0}-P_{0,2,1})$$

$$D_w(P(u,v,w))(0,1,0)=3(P_{1,2,0}-P_{0,3,0})$$

$$D_{uu}(P(u,v,w))(0,1,0)=6(P_{0,1,2}-2\mathrm{P}_{0,2,1}+P_{0,3,0})$$

$$D_{vv}(P(u,v,w))(0,1,0)=6(P_{2,1,0}-2\mathrm{P}_{1,1,1}+P_{0,1,2})$$

$$D_{ww}(P(u,v,w))(0,1,0)=6(P_{2,1,0}-2\mathrm{P}_{1,2,0}+P_{0,3,0})$$

Same at other corners

$$D_{uv}(P(u,v,w))(0,1,0)=6(P_{1,1,1}+P_{0,2,1}-P_{0,1,2}-P_{1,2,0})$$

$$D_{uw}(P(u,v,w))(0,1,0)=6(P_{1,1,1}+P_{0,3,0}-P_{0,2,1}-P_{1,2,0})$$

$$D_{vw}(P(u,v,w))(0,1,0)=6(P_{2,1,0}+P_{0,2,1}-P_{1,2,0}-P_{1,1,1})$$

NURBS surfaces

Definition and properties

- NURBS surfaces
  - As for rational curves, a weight is used (homogeneous coordinates)

$$P_{ij}^{w}=\begin{pmatrix} P_{ij}w_{ij} \\ w_{ij} \end{pmatrix}=\begin{pmatrix} x_{ij}w_{ij} \\ y_{ij}w_{ij} \\ z_{ij}w_{ij} \\ w_{ij} \end{pmatrix}$$

$$S^{w}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i}^{p}(u)N_{j}^{q}(v)P_{ij}^{w}$$

$$U=\{\underbrace{0,\cdots,0}_{p+1},u_{p+1},\cdots,u_{r-p-1},\underbrace{1,\cdots,1}_{p+1}\} \quad (r+1 \text{ nodes}, r=n+p+1)$$

$$V=\{\underbrace{0,\cdots,0}_{q+1},v_{q+1},\cdots,v_{s-q-1},\underbrace{1,\cdots,1}_{q+1}\} \quad (s+1 \text{ nodes}, s=m+q+1)$$

  - We have the equivalence $[wx,wy,wy,w]\equiv[x,y,z,1]$ $\forall w>0$

$$S(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}N_{i}^{p}(u)N_{j}^{q}(v)P_{ij}w_{ij}}{\sum_{k=0}^{n}\sum_{l=0}^{m}N_{k}^{p}(u)N_{l}^{q}(v)w_{kl}}$$

- NURBS Basis functions

$$S(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}N_i^p(u)N_j^q(v)P_{ij}w_{ij}}{\sum_{k=0}^{n}\sum_{l=0}^{m}N_k^p(u)N_l^q(v)w_{kl}}\equiv\sum_{i=0}^{n}\sum_{j=0}^{m}R_{ij}^{p,q}(u,v)P_{ij}$$

$$R_{ij}^{p,q}(u,v)=\frac{N_i^p(u)N_j^q(v)w_{ij}}{\sum_{k=0}^{n}\sum_{l=0}^{m}N_k^p(u)N_l^q(v)w_{kl}}$$

- Properties of basis functions of 3D NURBS
  - Partition of unity

$$\sum_{i=0}^{n}\sum_{j=0}^{m} R_{ij}^{p,q}(u,v)=1$$

  - Positivity (if the w are positive)

$$R_{ij}^{p,q}(u,v)\geq 0$$

  - NO property of tensor product

$$R_{ij}^{p,q}(u,v)\neq R_i^p(u)\,R_j^q(v)$$

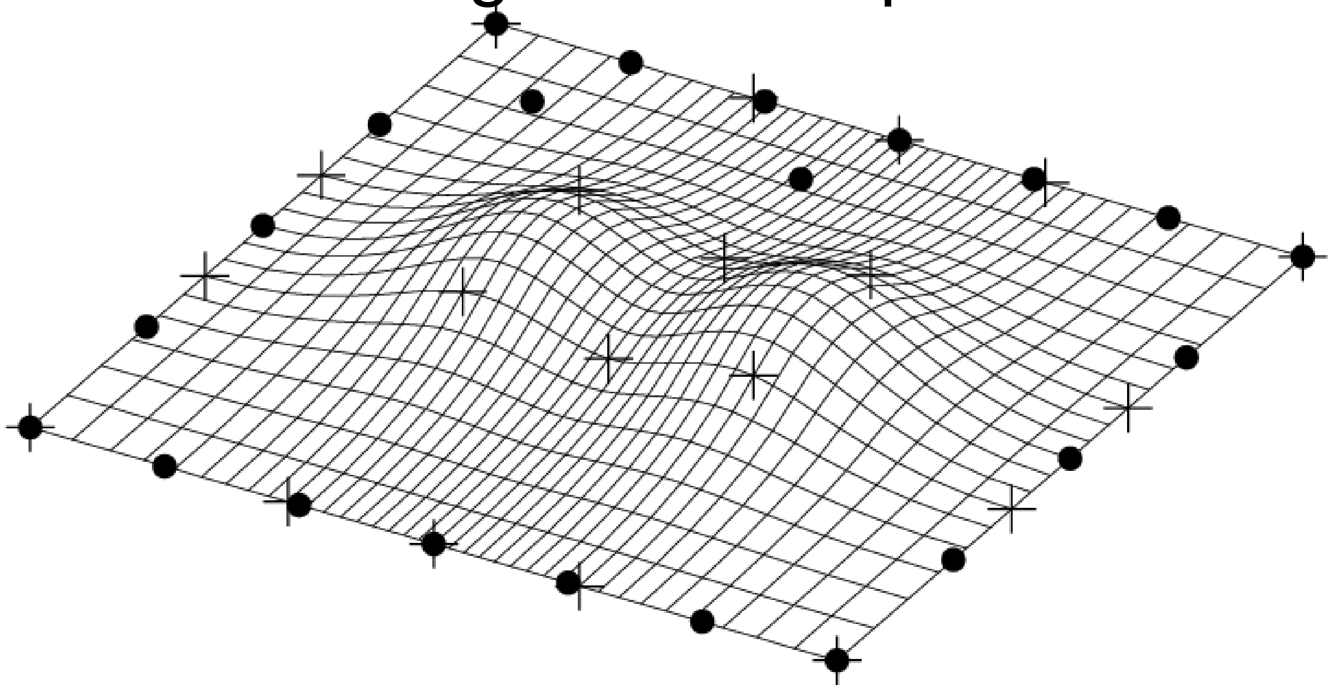
except if the weights are all equal

$$w_{ij}=a\neq 0\;\forall (i,j)\in\{0\cdots n\}\times\{0\cdots m\}$$

$$R_{ij}^{p,q}(u,v)=N_i^p(u)\,N_j^q(v)$$

- Properties of basis functions of 3D NURBS
  - Compact support
    $R_{ij}^{p,q}(u,v)=0 \quad \mathrm{si}(u,v)\notin[u_i,u_{i+p+1}[\times[v_j,v_{j+q+1}[$
    At the most $(p+1)(q+1)$ non zero basis functions on an interval $[u_{i0},u_{i0+1}[\times[v_{j0},v_{j0+1}[$:
    $R_{ij}^{p,q}(u,v)\neq 0 \quad \mathrm{si}(i,j)\in\{i0-p\cdots i0\}\times\{j0-q\cdots j0\}$
  - $R_{ij}^{p,q}(u,v)$ has exactly a maximum if $p>0$ et $q>0$.
  - If the nodal sequences with $p+1$ and $q+1$ repetitions :
    $R_{00}^{p,q}(u,v)=1, \quad R_{n0}^{p,q}(u,v)=1, \quad R_{0\mathrm{m}}^{p,q}(u,v)=1, \quad R_{nm}^{p,q}(u,v)=1$
  - Infinitely differentiable functions for $u\notin\{u_i\}$ and $v\notin\{v_j\}$ otherwise they are $(p\text{-}k_u)$ times differentiable (direction u) and/or $(q\text{-}k_v)$ times differentiable (direction v)

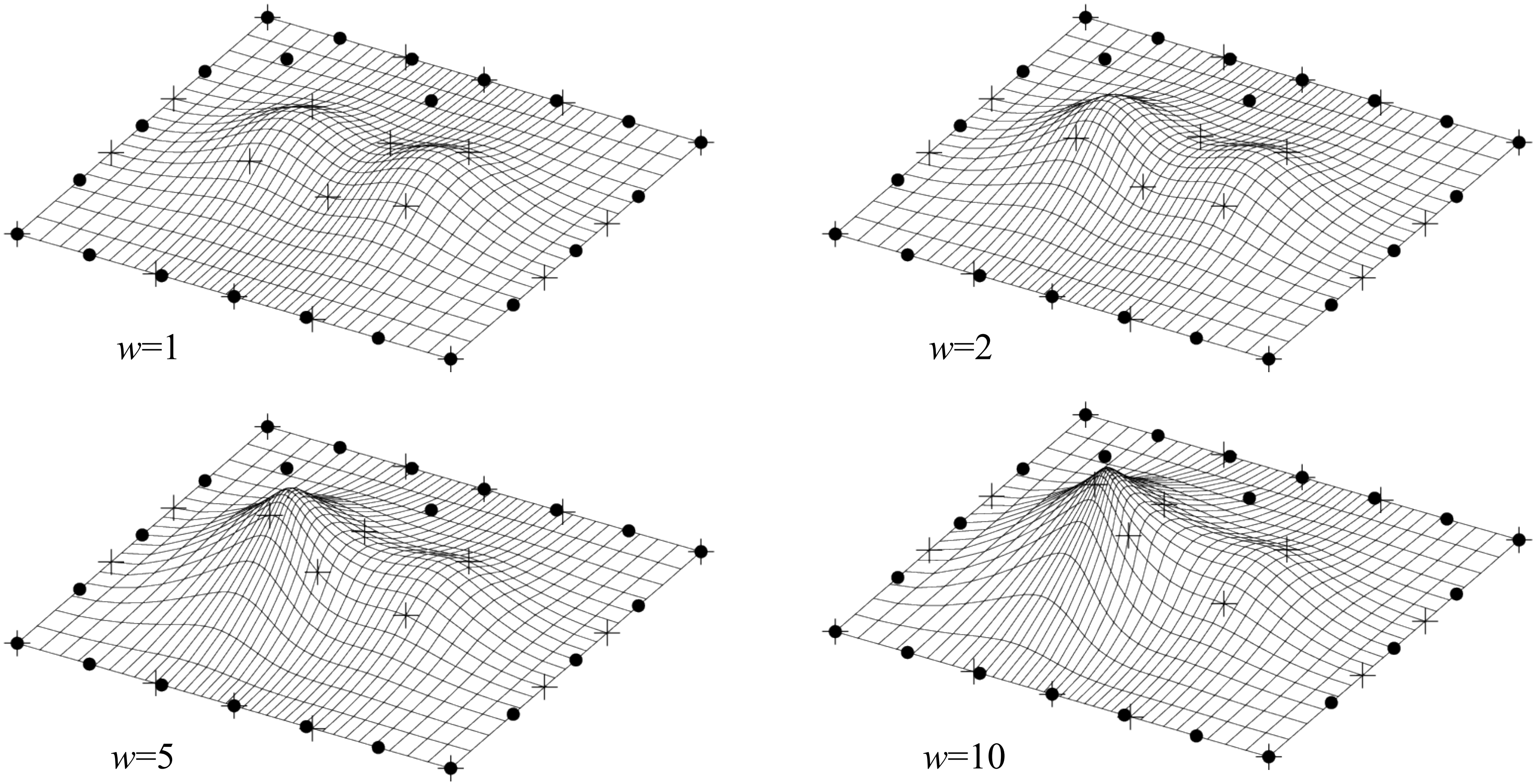
- Effect of the weight on a shape



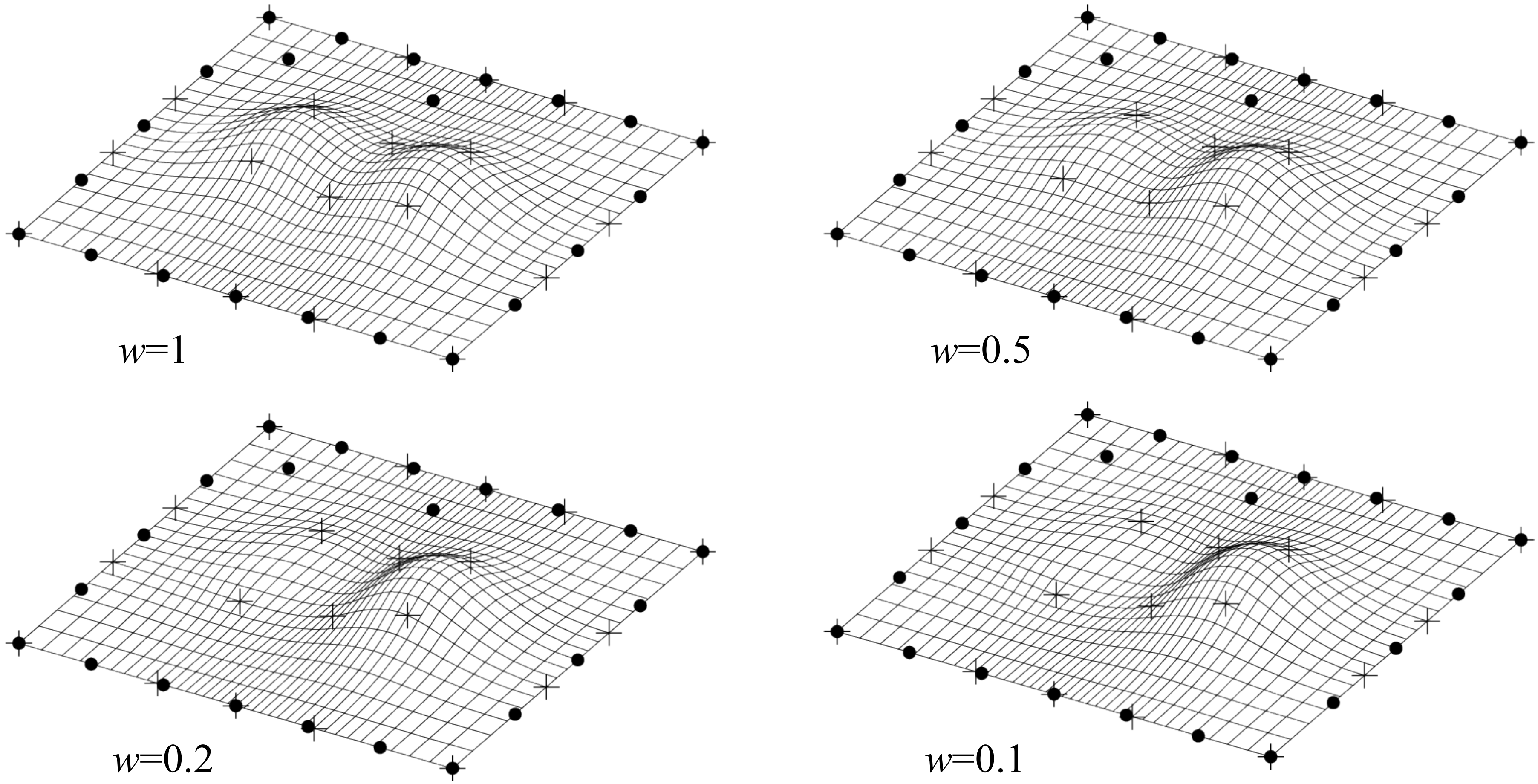
$$U=\{0,0,0,0,1,2,3,4,4,4,4\}$$

$$V=\{0,0,0,1,2,3,3,3\}$$

- Effect of the weight on a shape



- Effect of the weight on a shape



- Derivatives of a NURBS surface
  - It is easy to compute the derivatives in the space of homogeneous coordinates (4D) but ...we obviously want derivatives in the 3D space...

$$S(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}N_i^p(u)N_j^q(v)P_{ij}w_{ij}}{\sum_{k=0}^{n}\sum_{l=0}^{m}N_k^p(u)N_l^q(v)w_{kl}}$$

$$S(u,v)=\frac{A(u,v)}{w(u,v)}$$

$$\frac{\partial S(u,v)}{\partial\alpha}=\frac{\frac{\partial A(u,v)}{\partial\alpha}-\frac{\partial w(u,v)}{\partial\alpha}S(u,v)}{w(u,v)}\quad,\quad \alpha=u \text{ or } v$$

NURBS surfaces

- With a recursion, it is possible to show that :

$$\frac{\partial^{k+l} S}{\partial u^k \partial v^l} = \frac{1}{w}\left(\frac{\partial^{k+l} A}{\partial u^k \partial v^l} - \sum_{i=1}^{k} \binom{k}{i} \frac{\partial^i w}{\partial u^i} \frac{\partial^{(k-i)+l} S}{\partial u^{k-i} \partial v^l} - \sum_{j=1}^{l} \binom{l}{j} \frac{\partial^j w}{\partial v^j} \frac{\partial^{k+(l-j)} S}{\partial u^k \partial v^{l-j}} + \sum_{i=1}^{k} \binom{k}{i} \sum_{j=1}^{l} \binom{l}{j} \frac{\partial^{i+j} w}{\partial u^i \partial v^j} \frac{\partial^{(k-i)+(l-j)} S}{\partial u^{k-i} \partial v^{l-j}} \right)$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Second derivative

$$\frac{\partial^2 S}{\partial u \partial v} = \frac{1}{w}\left(\frac{\partial^2 A}{\partial u \partial v} - \frac{\partial^2 w}{\partial u \partial v} S - \frac{\partial w}{\partial u}\frac{\partial S}{\partial v} - \frac{\partial w}{\partial v}\frac{\partial S}{\partial u}\right)$$

$$\frac{\partial^2 S}{\partial u^2} = \frac{1}{w}\left(\frac{\partial^2 A}{\partial u^2} - \frac{\partial^2 w}{\partial u^2} S - 2\frac{\partial w}{\partial u}\frac{\partial S}{\partial u}\right)$$

$$\frac{\partial^2 S}{\partial v^2} = \frac{1}{w}\left(\frac{\partial^2 A}{\partial v^2} - \frac{\partial^2 w}{\partial v^2} S - 2\frac{\partial w}{\partial v}\frac{\partial S}{\partial v}\right)$$

Subdivision surfaces

Subdivision surfaces

- Parametric surfaces: an explicit representation
  - Lightweight
  - Discretization algorithms are non trivial... but it is necessary for display purposes and in computer graphics
  - Generally, these surfaces are used in cases where the geometric accuracy is essential, as in the computation of intersections and other precise geometric primitives
  - **Modelling operators are not trivial**
  - In computer graphics, such accuracy is generally not needed.

Subdivision surfaces

- Subdivision surfaces
  - Modelling basis = elementary mesh
  - By successive iterations, this mesh is refined up to the accuracy needed for the application
  - It is more like an algorithmic description vs. an algebraic representation, because the algorithm that is used to subdivide the mesh determines the final shape and the properties of the limiting surface (*ie.* when the number of subdivisions tends to the infinite)
  - Some of these limiting surfaces are equivalent to “regular” parametric surfaces, therefore have the same “accuracy”.

Subdivision surfaces

- History
  - 1974 – George Chaikin

    An algorithm for high speed curve generation
  - 1978 – Daniel Doo & Malcolm Sabin

    (D) A subdivision algorithm for smoothing irregularly shaped polyhedrons
    (D&S) Behaviour of recursive division surfaces near extraordinary points.
  - 1978 – Edwin Catmull & Jim Clark

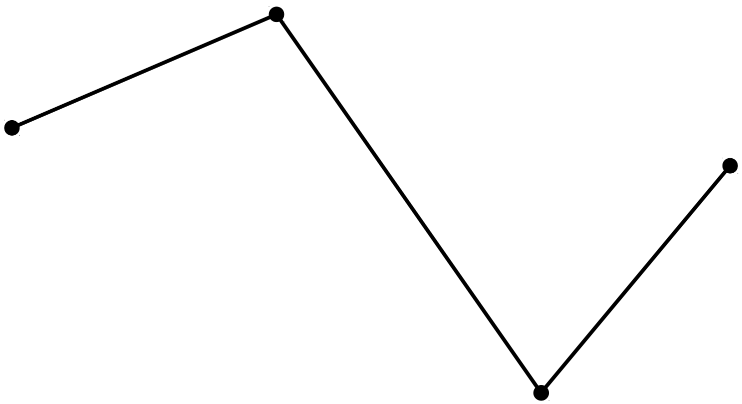
    Recursively generated B-Spline surfaces on arbitrary topological meshes
  - 1987 – Charles Loop

    Smooth subdivision surfaces based on triangles
  - 2000 – Leif Kobbelt

    $\sqrt{3}$ – subdivision (interpolating scheme)

Subdivision surfaces

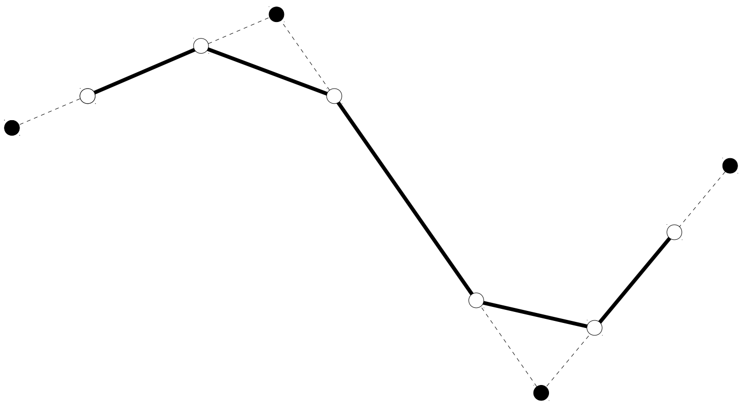
- Chaikin's scheme



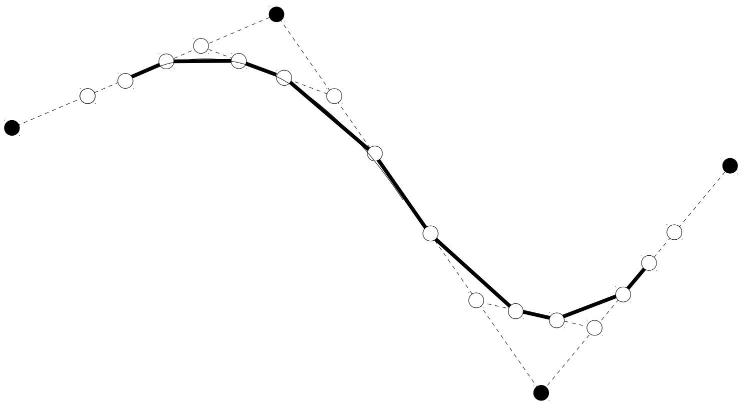
George Chaikin, An algorithm for high speed curve generation, *Computer graphics and Image Processing* **3** (1974) , 346-349

- Chaikin's scheme

  or « Corner-cutting »



- Chaikin's scheme

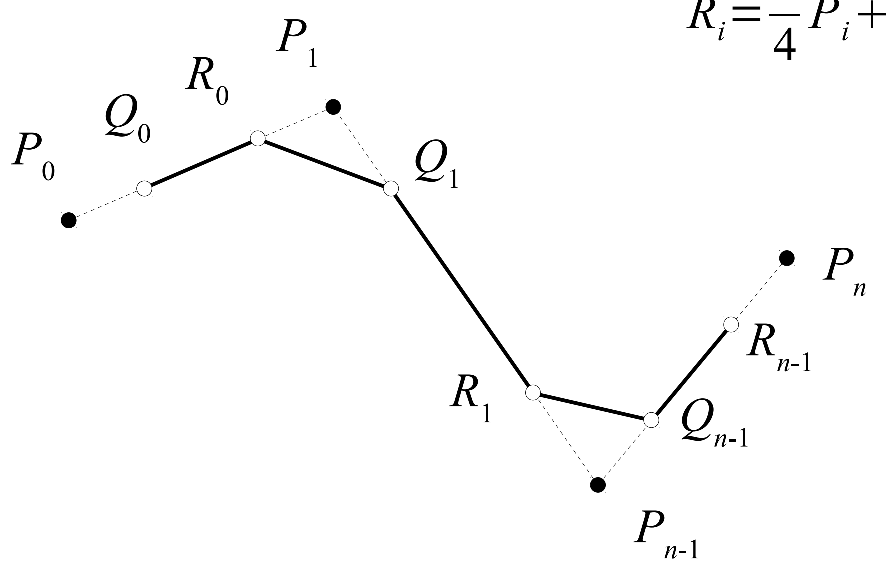


Chaikin's idea was simple : repeating the corner cutting, to the limit, one obtains a smooth curve

- Chaikin's scheme
  - Starting from a polygon having n vertices $\{P_0, P_1, \ldots P_{n-1}\}$, one builds the polygon having $2n$ vertices $\{Q_0, R_0, Q_1, R_1, \ldots Q_{n-1}, R_{n-1}\}$. This polygon serves as a basis for the next step of the algorithm : $\{P'_0, P'_1, \ldots P'_{2n-1}\}$
  - The new vertices are :

$$Q_i = \frac{3}{4} P_i + \frac{1}{4} P_{i+1}$$

$$R_i = \frac{1}{4} P_i + \frac{3}{4} P_{i+1}$$

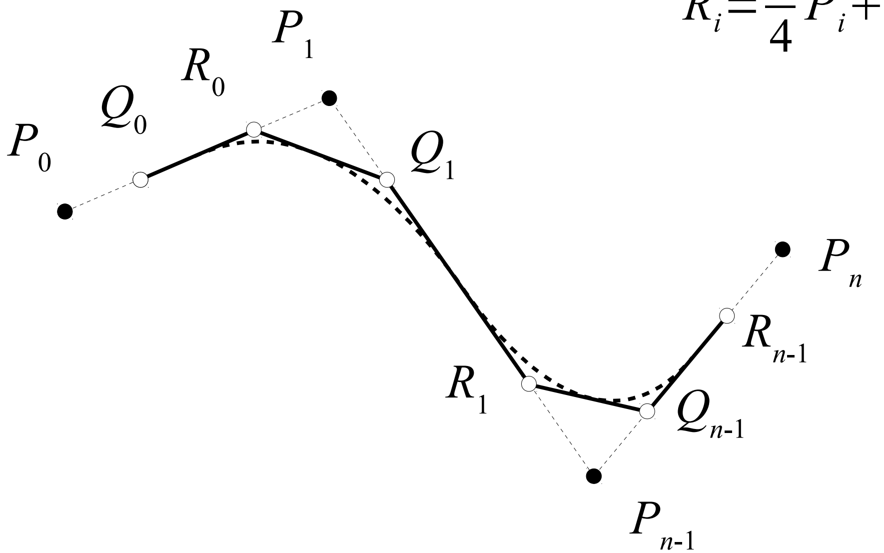


Subdivision surfaces

- Chaikin's scheme
  - Riesenfeld (1978) has shown that this algorithm leads at the limit to an uniform quadratic B-spline, which exhibits a C^1 continuity.

$$Q_i = \frac{3}{4} P_i + \frac{1}{4} P_{i+1}$$

$$R_i = \frac{1}{4} P_i + \frac{3}{4} P_{i+1}$$



Subdivision surfaces

- Demonstration of the equivalence of Chaikin's scheme and uniform quadratic B-Splines
  - The B-Spline curve is defined by : $P(u)=\sum_{i=0}^{n} P_i N_i^2(u)$

$$N_i^2(u)=\frac{u-u_i}{u_{i+2}-u_i} N_i^1(u)+\frac{u_{i+3}-u}{u_{i+3}-u_{i+1}} N_{i+1}^1(u)$$

$$N_i^1(u)=\frac{u-u_i}{u_{i+1}-u_i} N_i^0(u)+\frac{u_{i+2}-u}{u_{i+2}-u_{i+1}} N_{i+1}^0(u)$$

$$N_i^0(u)=\begin{cases} 1 & \text{if} \quad u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

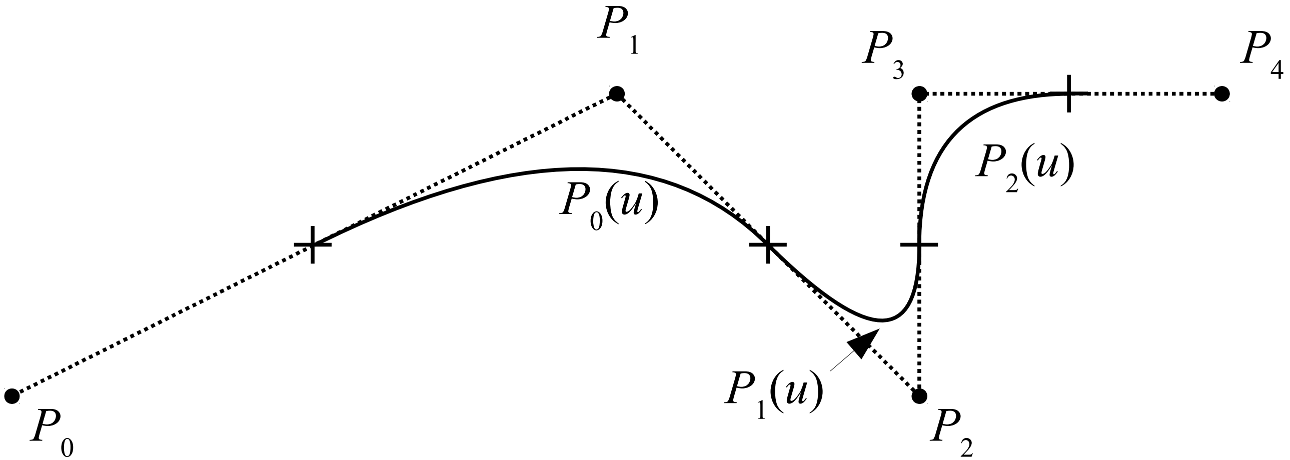
$$U=\{u_0,\cdots,u_{n+2}\} \ , \ u_{i+1}-u_i=1 \ , \ i=0\cdots n+1$$

- It can be rewritten as an assembly of curve “parts” :

$$P(u)=\sum_{i=0}^{n} P_i N_i^2(u)=\sum_{k=0}^{n-2} P_k(u)$$ « Portion » of curve

with $$P_k(u)=\begin{bmatrix}1 & u & u^2\end{bmatrix}\cdot M_k\cdot\begin{bmatrix}P_k\\ P_{k+1}\\ P_{k+2}\end{bmatrix}$$

- The matrix M_k depends on the nodal sequence U.



Subdivision surfaces

- Computation of shape functions of degree $d \leq 2$ for $u_2 = 0 \leq u \leq u_3 = 1$

$$U = \{u_0 = -2, u_1 = -1, u_2 = 0, u_3 = 1, u_4 = 2, u_5 = 3\}$$

$$N_0^0 = 0 \quad N_1^0 = 0 \quad N_2^0 = 1 \quad N_3^0 = 0 \quad N_4^0 = 0$$

$$N_0^1 = 0 \quad N_1^1 = 1 - u \quad N_2^1 = u \quad N_3^1 = 0$$

$$N_0^2 = \frac{1}{2}(1 - 2u + u^2)$$

$$N_1^2 = \frac{1}{2}(1 + 2u - 2u^2)$$

$$N_2^2 = \frac{1}{2}(u^2)$$

Therefore, $M_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

General case : $M_k = \dfrac{1}{u_{k+3} - u_{k+2}} \begin{bmatrix} \dfrac{u_{k+3}^2}{\alpha} & -\dfrac{u_{k+3}u_{k+1}}{\alpha} - \dfrac{u_{k+4}u_{k+2}}{\beta} & \dfrac{u_{k+2}^2}{\beta} \\ -2\dfrac{u_{k+3}}{\alpha} & \dfrac{u_{k+3}+u_{k+1}}{\alpha} + \dfrac{u_{k+4}+u_{k+2}}{\beta} & -2\dfrac{u_{k+2}}{\beta} \\ \dfrac{1}{\alpha} & -\dfrac{1}{\alpha} - \dfrac{1}{\beta} & \dfrac{1}{\beta} \end{bmatrix}$

with : $\alpha = u_{k+3} - u_{k+1}$

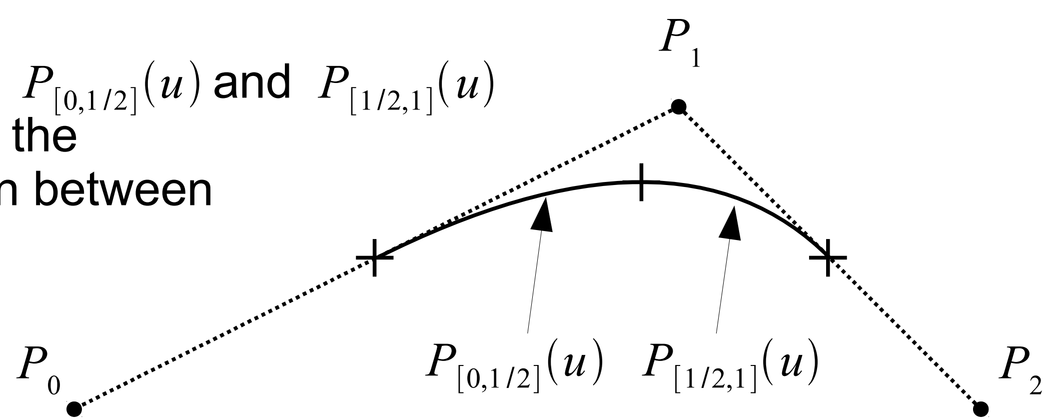
$\beta = u_{k+4} - u_{k+2}$

Subdivision surfaces

- Binary subdivision of a B-Spline curve for $0 \leq u \leq 1$
  - One has to find the new set of control points for each halves of the curve
  - We set $n=2$ (number of control points -1)

$$P(u) = [1 \quad u \quad u^2] \cdot M \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \qquad M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

  - One wants to express $P_{[0,1/2]}(u)$ and $P_{[1/2,1]}(u)$ - on each subdivision, the parameter u shall be in between 0 and 1.



- Case of $P_{[0,1/2]}(u)$

$$P_{[0,1/2]}(u) = P(u/2) = [1 \quad u/2 \quad u^2/4] \cdot M \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

$$= [1 \quad u \quad u^2] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \cdot M \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

$$= [1 \quad u \quad u^2] \cdot M \cdot M^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \cdot M \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

$$= [1 \quad u \quad u^2] \cdot M \cdot \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} \quad \text{avec} \quad \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \underbrace{M^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \cdot M}_{S_{[0,1/2]}} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

- Case of $P_{[1/2,1]}(u)$

$$P_{[1/2,1]}(u)=P((1+u)/2)=\begin{bmatrix}1 & (1+u)/2 & (1+u)^2/4\end{bmatrix}\cdot M\cdot\begin{bmatrix}P_0\\P_1\\P_2\end{bmatrix}$$

$$=\begin{bmatrix}1 & u & u^2\end{bmatrix}\cdot\begin{bmatrix}1 & 1/2 & 1/4\\0 & 1/2 & 1/2\\0 & 0 & 1/4\end{bmatrix}\cdot M\cdot\begin{bmatrix}P_0\\P_1\\P_2\end{bmatrix}$$

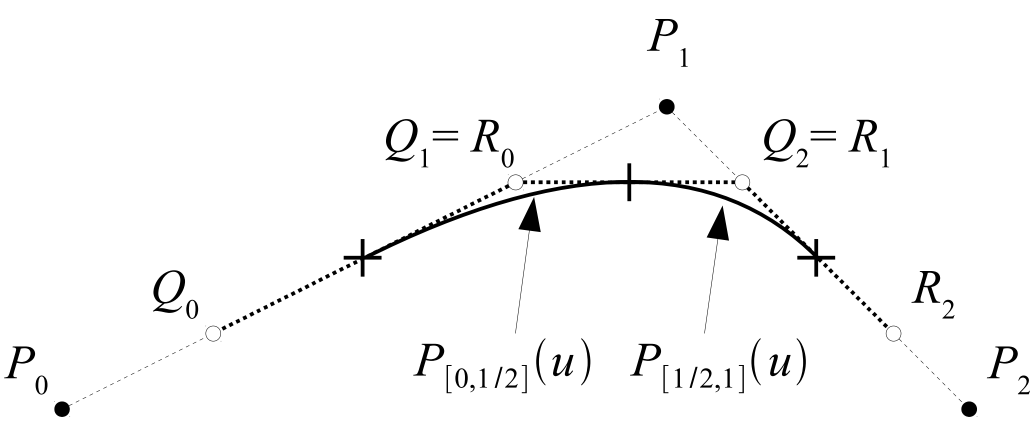
$$=\begin{bmatrix}1 & u & u^2\end{bmatrix}\cdot M\cdot M^{-1}\cdot\begin{bmatrix}1 & 1/2 & 1/4\\0 & 1/2 & 1/2\\0 & 0 & 1/4\end{bmatrix}\cdot M\cdot\begin{bmatrix}P_0\\P_1\\P_2\end{bmatrix}$$

$$=\begin{bmatrix}1 & u & u^2\end{bmatrix}\cdot M\cdot\begin{bmatrix}R_0\\R_1\\R_2\end{bmatrix}\quad\text{avec}\quad\begin{bmatrix}R_0\\R_1\\R_2\end{bmatrix}=\underbrace{M^{-1}\cdot\begin{bmatrix}1 & 1/2 & 1/4\\0 & 1/2 & 1/2\\0 & 0 & 1/4\end{bmatrix}\cdot M}_{S_{[1/2,1]}}\cdot\begin{bmatrix}P_0\\P_1\\P_2\end{bmatrix}$$

- Finally,

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = S_{[0,1/2]} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \qquad S_{[0,1/2]} = M^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \cdot M = \frac{1}{4} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3P_0 + P_1 \\ P_0 + 3P_1 \\ 3P_1 + P_2 \end{bmatrix}$$

$$\begin{bmatrix} R_0 \\ R_1 \\ R_2 \end{bmatrix} = S_{[1/2,1]} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \qquad S_{[1/2,1]} = M^{-1} \cdot \begin{bmatrix} 1 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/4 \end{bmatrix} \cdot M = \frac{1}{4} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} R_0 \\ R_1 \\ R_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} P_0 + 3P_1 \\ 3P_1 + P_2 \\ P_1 + 3P_2 \end{bmatrix}$$



One finds the same coefficients as in Chaikin's scheme... (except for the indices)

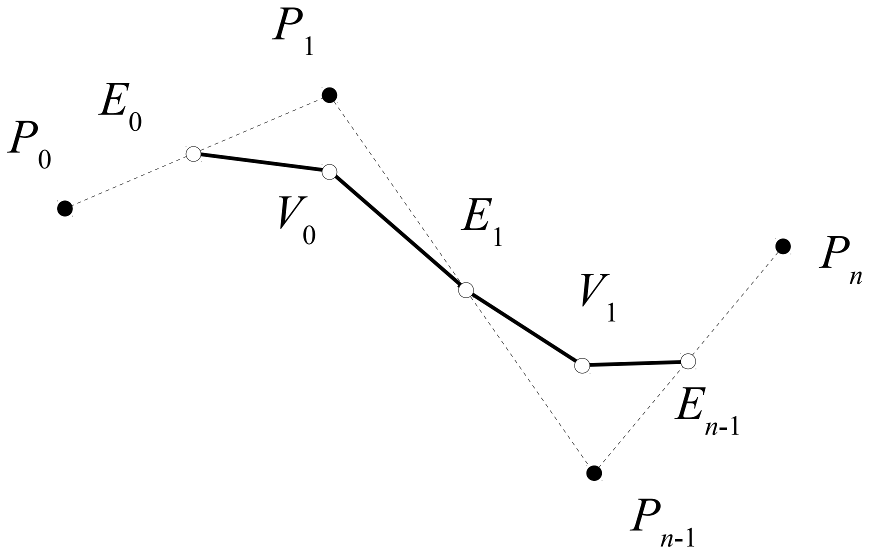
$$Q_i = \frac{3}{4} P_i + \frac{1}{4} P_{i+1}$$

$$R_i = \frac{1}{4} P_i + \frac{3}{4} P_{i+1}$$

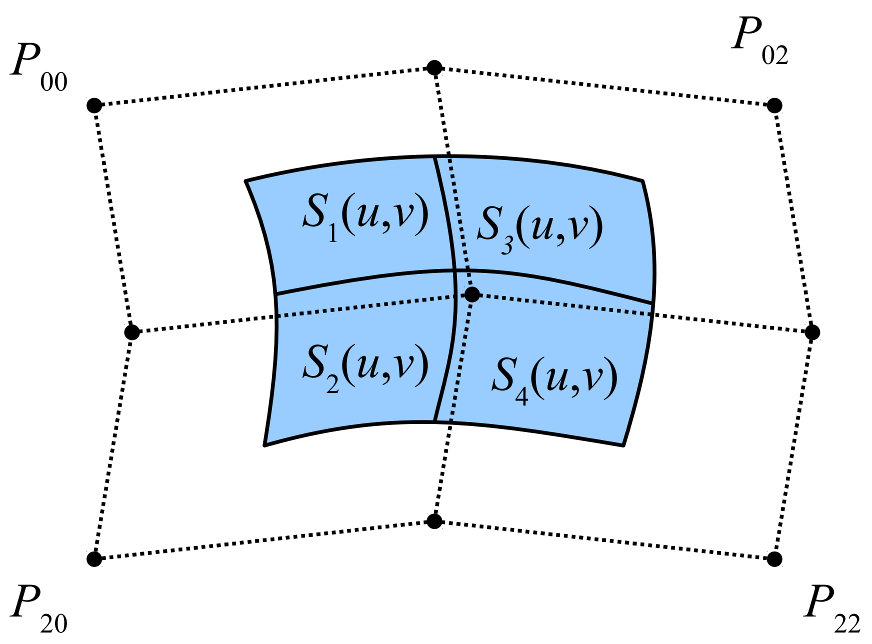
- This can be extended to cubic B-Splines
  - C^2 continuity

$$E_i = \frac{1}{2} P_i + \frac{1}{2} P_{i+1}$$

$$V_i = \frac{1}{8} P_i + \frac{3}{4} P_{i+1} + \frac{1}{8} P_{i+2}$$



- Doo-Sabin scheme
  - This is an extension of Chaikin's scheme for a uniform biquadratic B-Spline surface
  - The new mesh is built using the control points resulting from the subdivision of the original patch into 4 new sub-patches.



- Expression of the bi-quadratic patch as a monomial form for $0 \le u \le 1$ and $0 \le v \le 1$:

$$S(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2} N_i^2(u)\,N_j^2(v)\,P_{ij}$$

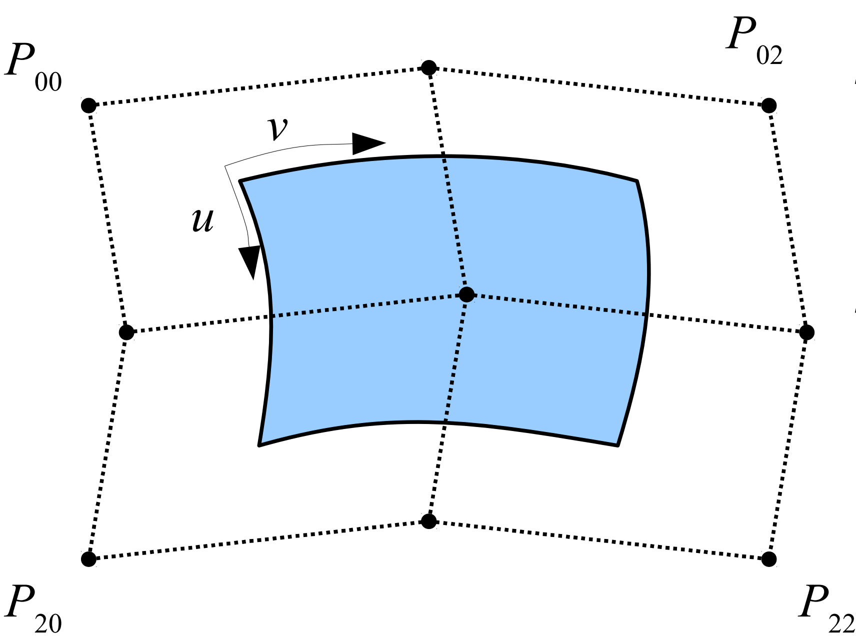
$$U=V=\{-2,-1,0,1,2,3\}$$

$$N_0^2(t)=\frac{1}{2}(1-2t+t^2)$$

$$N_1^2(t)=\frac{1}{2}(1+2t-2t^2)$$

(with $t=u$ or v)

$$N_2^2(t)=\frac{1}{2}(t^2)$$



$$S(u,v)=[1 \quad u \quad u^2]\cdot M\cdot\begin{bmatrix}P_0(v)\\P_1(v)\\P_2(v)\end{bmatrix}$$

$$S(u,v)=[1 \quad u \quad u^2]\cdot M\cdot\begin{bmatrix}P_{00} & P_{01} & P_{02}\\P_{10} & P_{11} & P_{12}\\P_{20} & P_{21} & P_{22}\end{bmatrix}\cdot M^{\mathrm{T}}\cdot\begin{bmatrix}1\\v\\v^2\end{bmatrix}$$

Again, $M=\frac{1}{2}\begin{bmatrix}1 & 1 & 0\\-2 & 2 & 0\\1 & -2 & 1\end{bmatrix}$

Subdivision surfaces

- Subdivision - patch $S_1(u,v)$

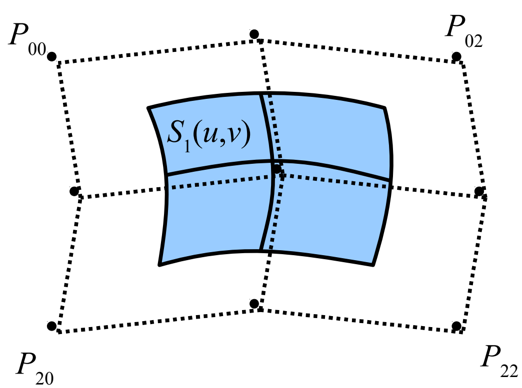
$$S_1(u,v)=S(u/2,v/2)=[1 \;\; u/2 \;\; u^2/4]\cdot M\cdot P\cdot M^T\cdot\begin{bmatrix}1\\ v/2\\ v^2/4\end{bmatrix} \qquad P=\begin{bmatrix}P_{00} & P_{01} & P_{02}\\ P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\end{bmatrix}$$

$$S_1(u,v)=S(u/2,v/2)=[1 \;\; u \;\; u^2]\cdot C\cdot M\cdot P\cdot M^T\cdot C^T\begin{bmatrix}1\\ v\\ v^2\end{bmatrix} \qquad C=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/4\end{bmatrix}$$

$$=[1 \;\; u \;\; u^2]\cdot M\cdot M^{-1}\cdot C\cdot M\cdot P\cdot M^T\cdot C^T\cdot(M^{-1})^T\cdot M^T\cdot\begin{bmatrix}1\\ v\\ v^2\end{bmatrix}$$

$$=[1 \;\; u \;\; u^2]\cdot M\cdot(M^{-1}\cdot C\cdot M)\cdot P\cdot(M^{-1}\cdot C\cdot M)^T\cdot M^T\cdot\begin{bmatrix}1\\ v\\ v^2\end{bmatrix}$$

$$=[1 \;\; u \;\; u^2]\cdot M\cdot P'\cdot M^T\cdot\begin{bmatrix}1\\ v\\ v^2\end{bmatrix} \qquad P'=S\cdot P\cdot S^T \qquad S=M^{-1}\cdot C\cdot M$$

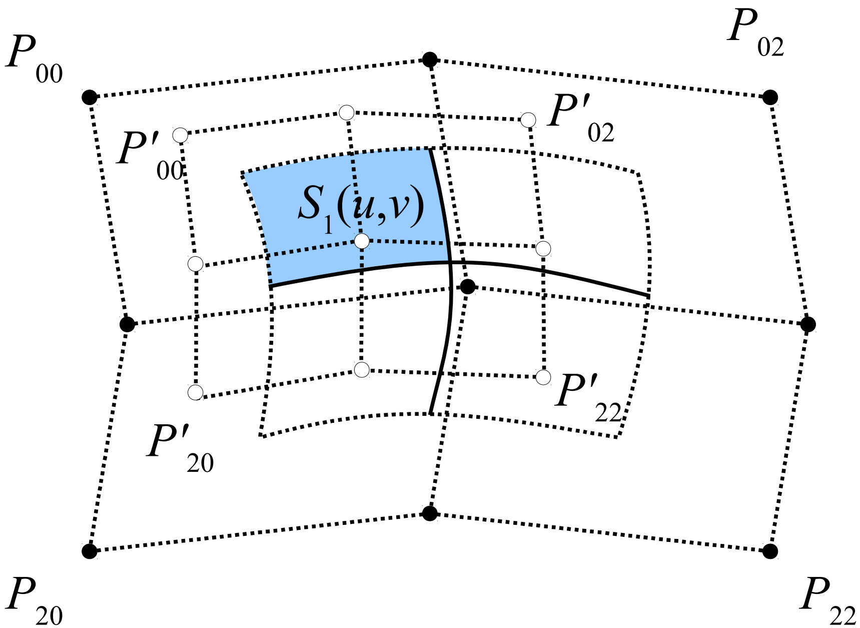


Subdivision surfaces

- Finally,

$$P' = S \cdot P \cdot S^{T} \quad S = M^{-1} \cdot C \cdot M = \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$P' = \frac{1}{16}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$



$$P' = \frac{1}{16}\begin{bmatrix} 3(3P_{00}+P_{10})+3P_{01}+P_{11} & 3P_{00}+P_{10}+3(3P_{01}+P_{11}) & 3(3P_{01}+P_{11})+3P_{02}+P_{12} \\ 3(P_{00}+3P_{10})+P_{01}+3P_{11} & P_{00}+3P_{10}+3(P_{01}+3P_{11}) & 3(P_{01}+3P_{11})+P_{02}+3P_{12} \\ 3(3P_{10}+P_{20})+3P_{11}+P_{21} & 3P_{10}+P_{20}+3(3P_{11}+P_{21}) & 3(3P_{11}+P_{21})+3P_{12}+P_{22} \end{bmatrix}$$

- Same developments should be done with the 3 other quadrants, and will lead to the same “structure”

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Subdivision surfaces

- Subdivision - patch $S_2(u,v)$

$$S_2(u,v)=S(u/2,(1+v)/2)=[1 \;\; u/2 \;\; u^2/4]\cdot M\cdot P\cdot M^{\mathrm{T}}\cdot\begin{bmatrix}1\\(1+v)/2\\(1+v)^2/4\end{bmatrix} \qquad P=\begin{bmatrix}P_{00} & P_{01} & P_{02}\\P_{10} & P_{11} & P_{12}\\P_{20} & P_{21} & P_{22}\end{bmatrix}$$

$$S_2(u,v)=S(u/2,(1+v)/2)=[1 \;\; u \;\; u^2]\cdot C_u\cdot M\cdot P\cdot M^{\mathrm{T}}\cdot C_v^{\mathrm{T}}\begin{bmatrix}1\\v\\v^2\end{bmatrix}$$

$$C_u=\begin{bmatrix}1 & 0 & 0\\0 & 1/2 & 0\\0 & 0 & 1/4\end{bmatrix} \qquad C_v=\begin{bmatrix}1 & 1/2 & 1/4\\0 & 1/2 & 1/2\\0 & 0 & 1/4\end{bmatrix}$$

$$=[1 \;\; u \;\; u^2]\cdot M\cdot(M^{-1}\cdot C_u\cdot M)\cdot P\cdot(M^{-1}\cdot C_v\cdot M)^{\mathrm{T}}\cdot M^{\mathrm{T}}\cdot\begin{bmatrix}1\\v\\v^2\end{bmatrix}$$

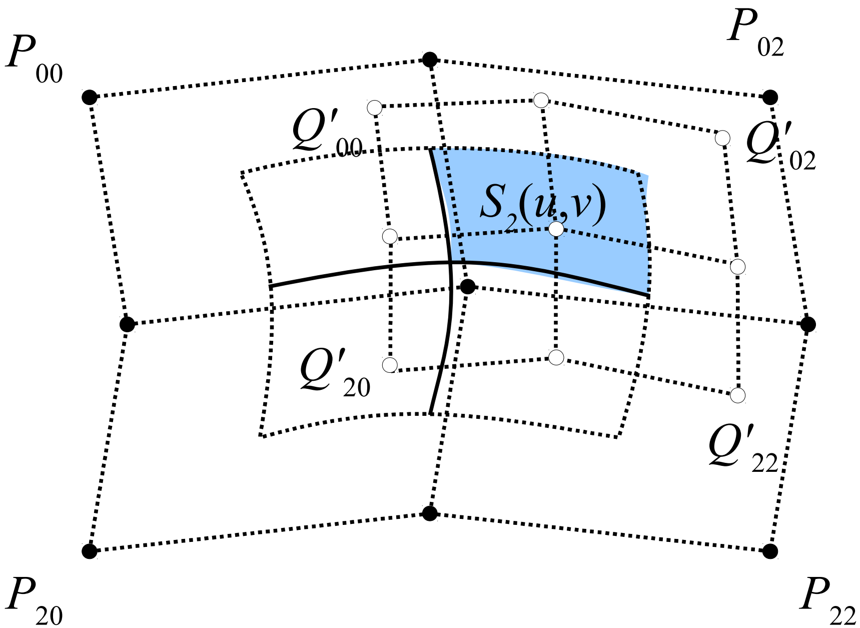
$$=[1 \;\; u \;\; u^2]\cdot M\cdot Q'\cdot M^{\mathrm{T}}\cdot\begin{bmatrix}1\\v\\v^2\end{bmatrix} \qquad Q'=S_u\cdot P\cdot S_v^{\mathrm{T}} \qquad \begin{aligned}S_u&=M^{-1}\cdot C_u\cdot M\\ S_v&=M^{-1}\cdot C_v\cdot M\end{aligned}$$

Subdivision surfaces

$$Q' = S_u \cdot P\, S_v^{\mathrm{T}} \qquad S_u = M^{-1} \cdot C_u \cdot M = \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$S_v = M^{-1} \cdot C_v \cdot M = \frac{1}{4}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$Q' = \frac{1}{16}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$



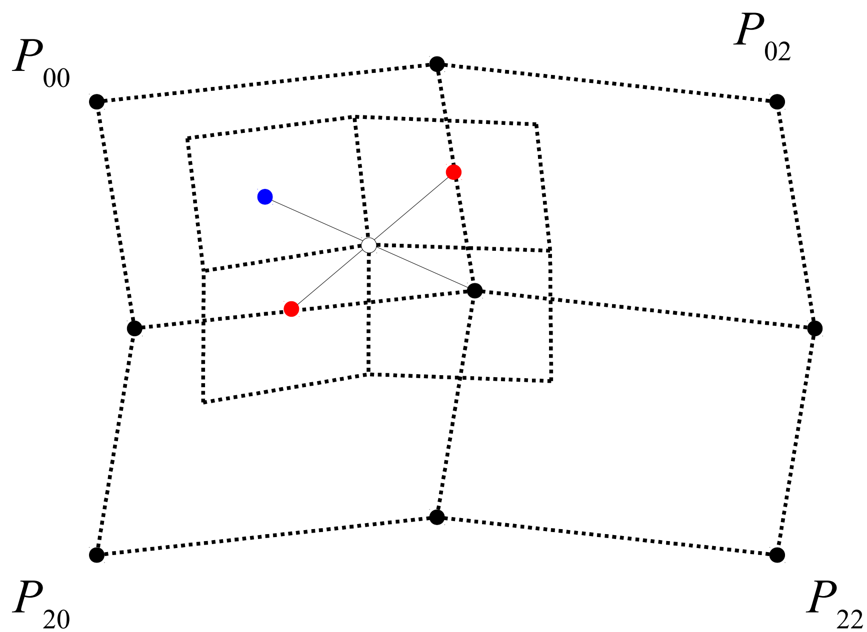
- Some of the points (2) are already computed, e.g.

$$Q'_{00} = 3\left(P_{00} + 3P_{01}\right) + P_{10} + 3P_{11}$$

$$\left(= P'_{01} = 3P_{00} + P_{10} + 3\left(3P_{01} + P_{11}\right) \right)$$

Subdivision surfaces

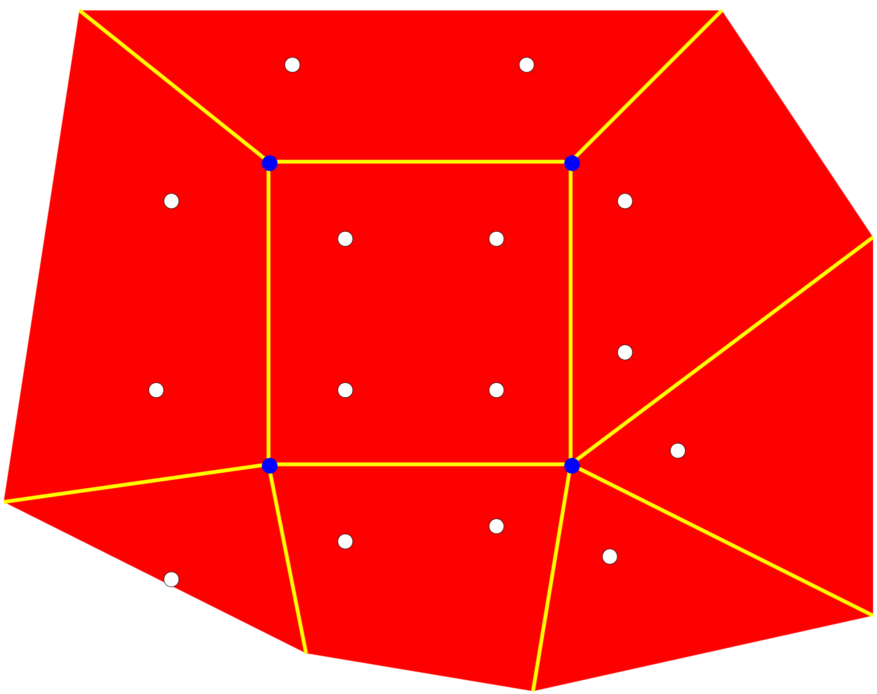
- Extension to meshes showing an arbitrary **topology**



- The new vertices are obtained as a simple arithmetic mean of 3 categories of vertices :
  - The vertices of the old mesh
  - Vertices on the edges (barycentre of the extremities of the edge)
  - Vertices inside a face (barycentre of the vertices of the face)

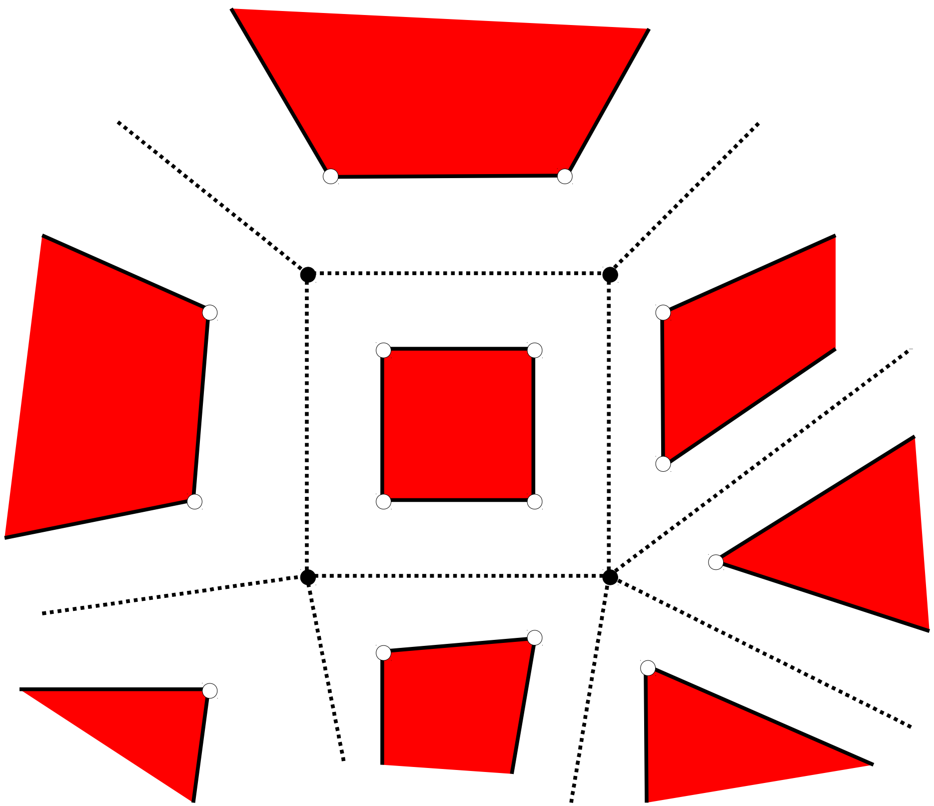
- Extension to meshes showing an arbitrary topology

  1 – Computation of the vertices P' (for each vertex P, compute the mean between P, the vertices on the adjacent faces, and the vertices on adjacent edges)



- Extension to meshes showing an arbitrary topology

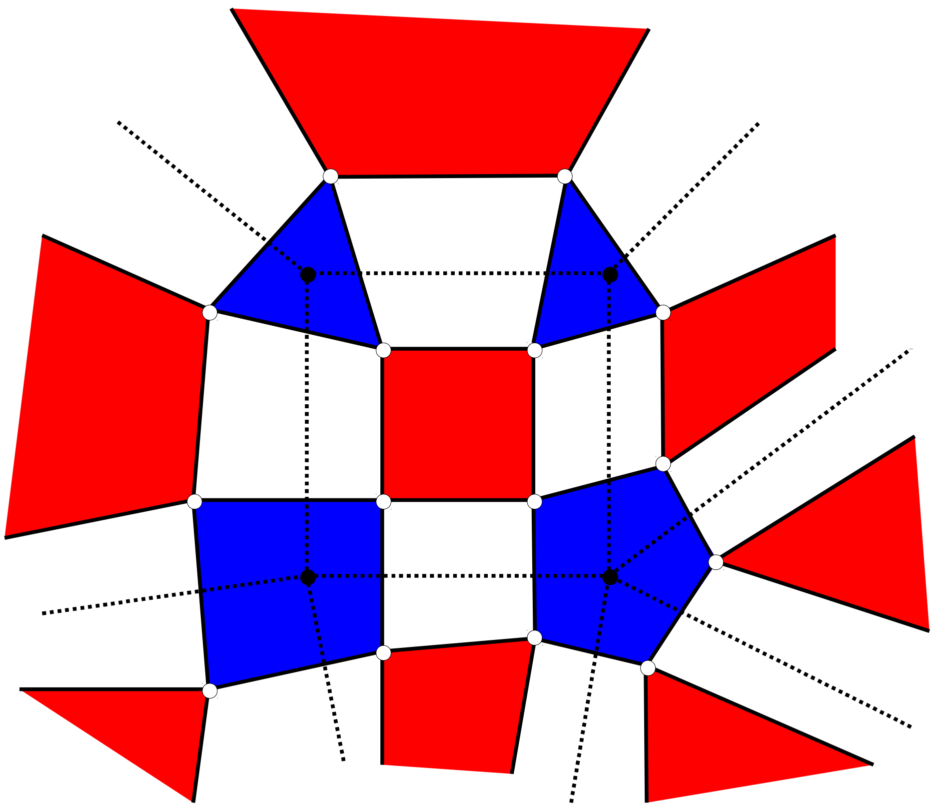
  2 – For each face, link the corresponding vertices P'



Subdivision surfaces

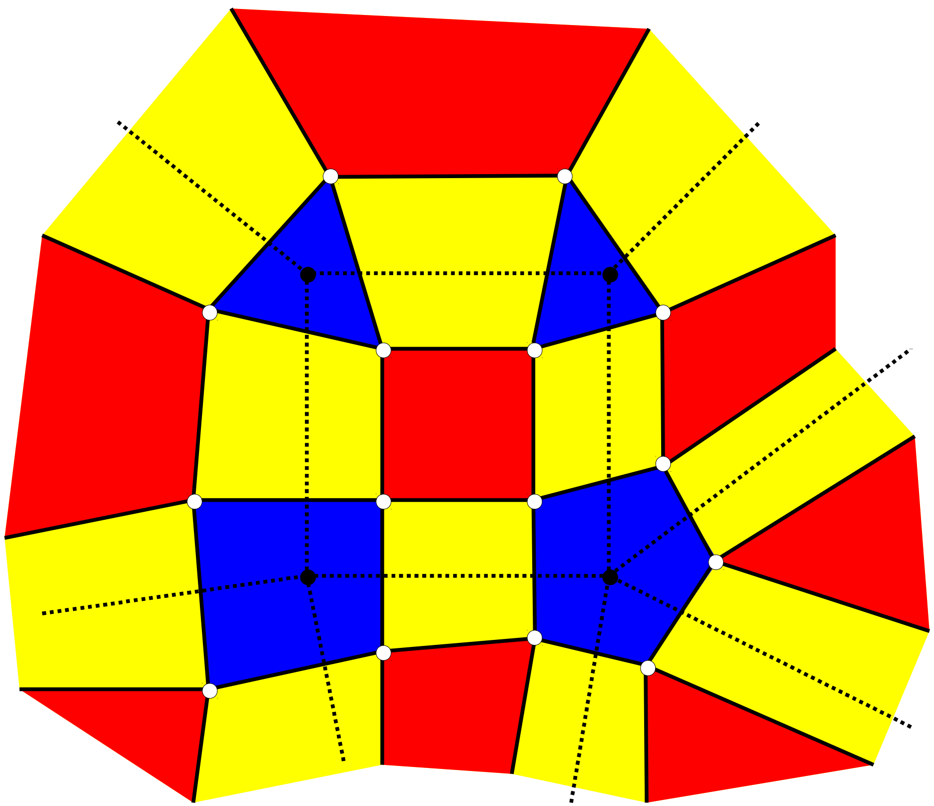
- Extension to meshes showing an arbitrary topology

  3 – For each old vertex, connect the new ones that have been created for each adjacent face to this old vertex.



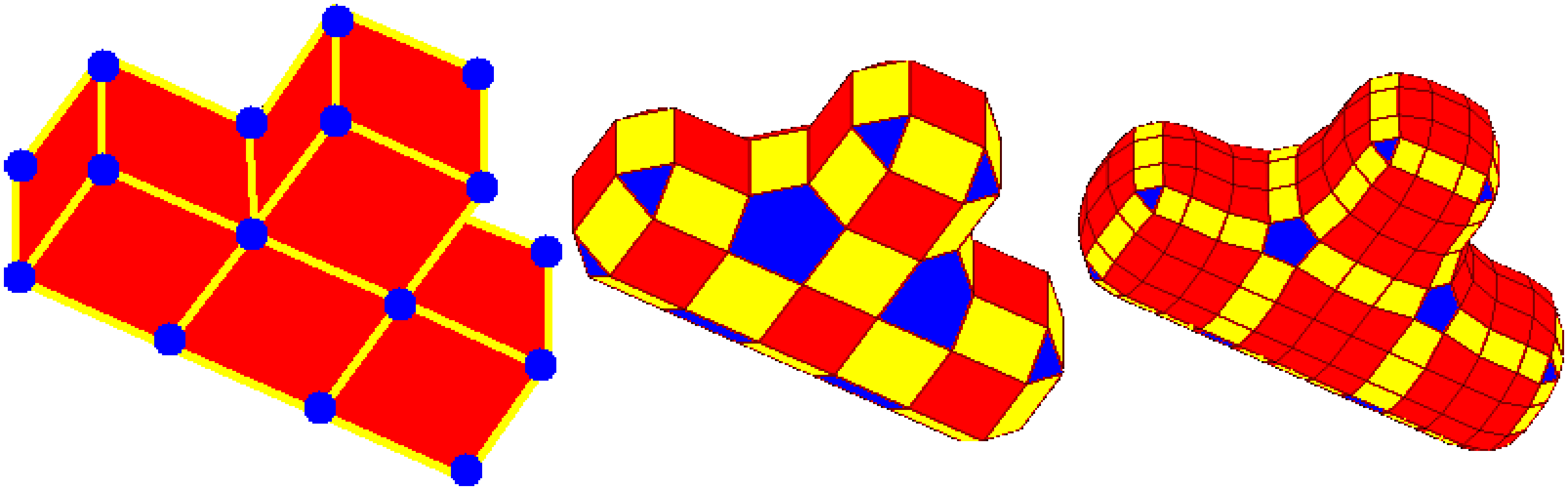
- Extension to meshes showing an arbitrary topology

  4 – For each old edge, connect the new vertices that have been created for each adjacent face to this old edge.



Subdivision surfaces

- Some points on the mesh and the limiting surface are « extraordinary »
  - These are vertices with a valence (number of incident edges) that is different from 4.



  - Everywhere the continuity of the limiting surface is C^1 ; except at extraordinary points, where it decreases to C^0.

Image : wikipedia

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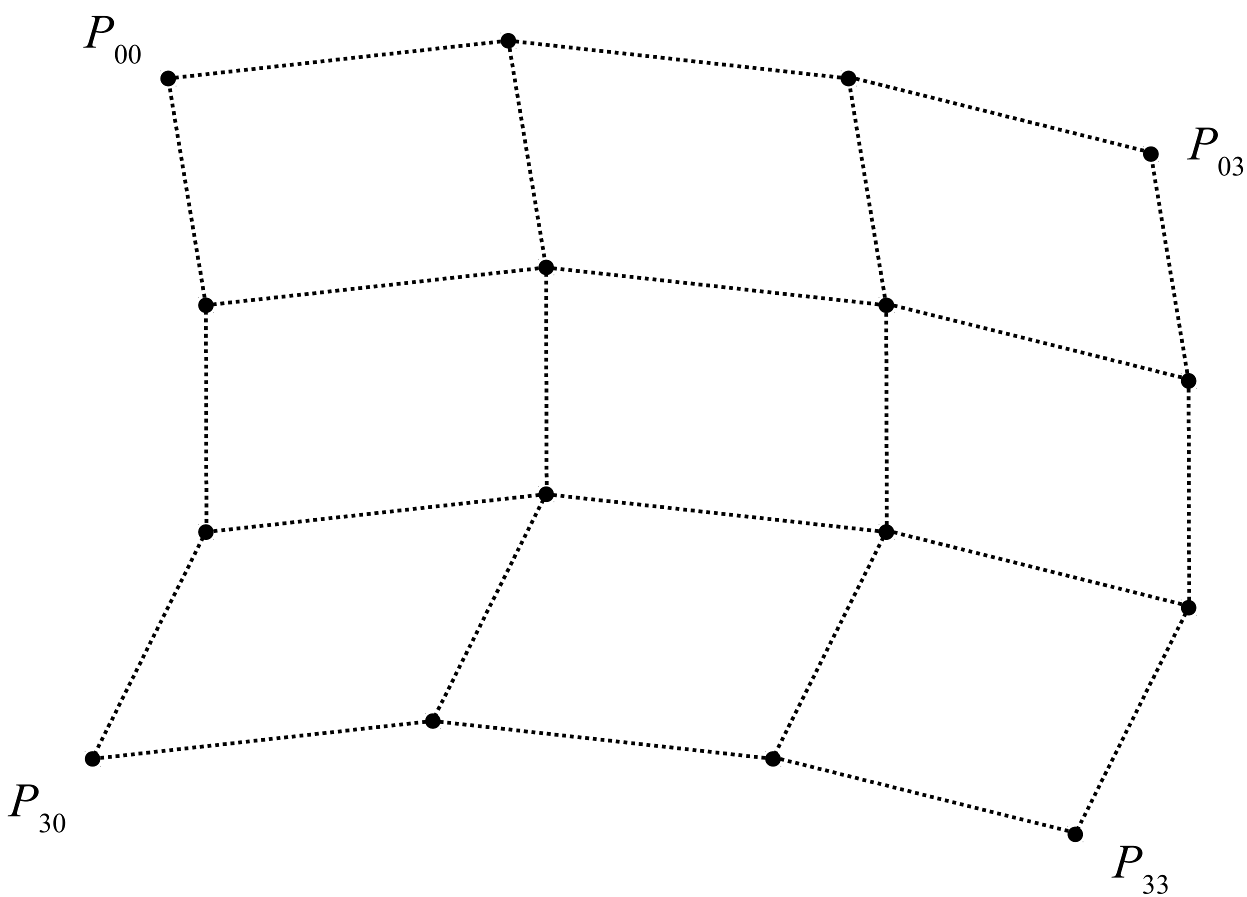
Subdivision surfaces

- Catmull-Clark scheme
  - Similar idea for bicubic B-Splines (proof : Jos Stam, Siggraph 1998)



Subdivision surfaces

- Three types of vertices

- « Face » vertices are at the barycentre of the vertices of that face:

$$P_f = Q$$

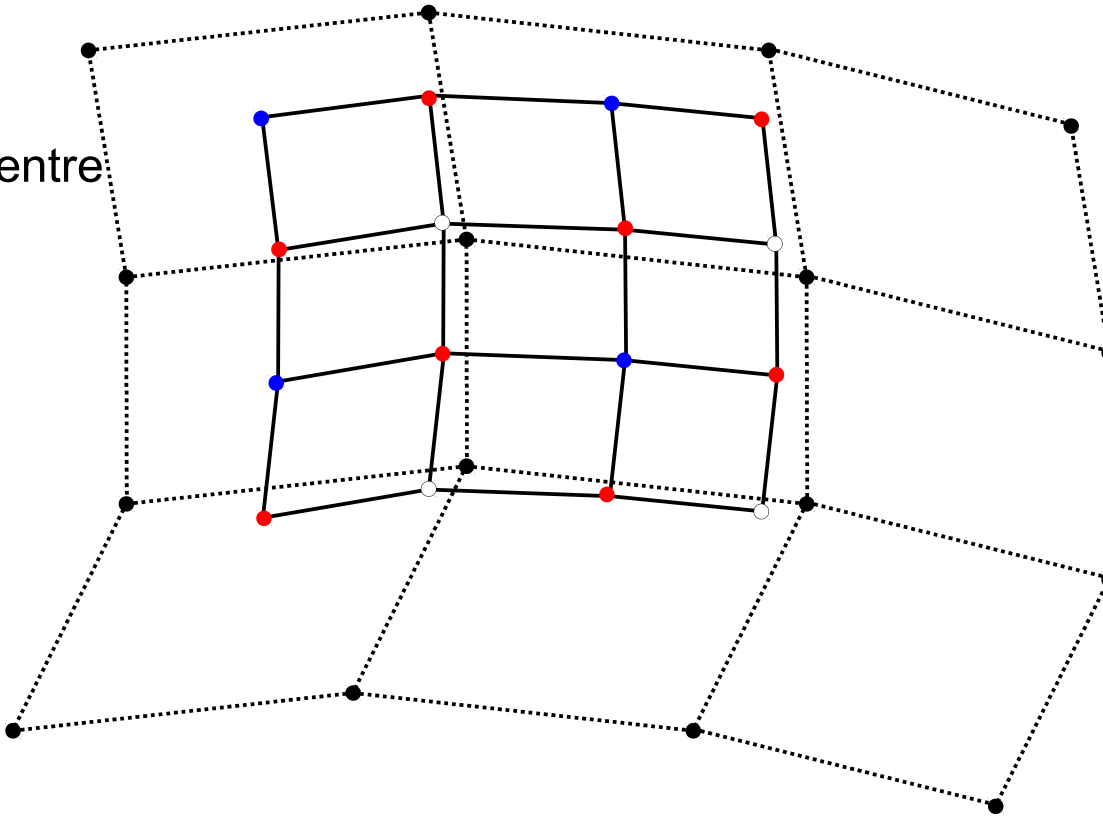
- « Edge » vertices are at the barycentre of the extremities of the edge and the two « Face » vertices of the adjacent faces :

$$P_e = \frac{Q+R}{2}$$

- « Corner » vertices are positioned such that :

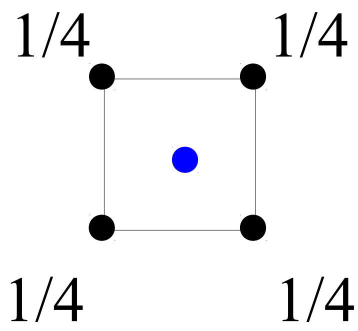
$$P_v = \frac{Q+2R+(n-3)S}{n}$$

Q = mean of the barycentre of the incident faces
R = mean of the barycentre of the incident edges
S = original vertex
n = number of incident edges to S *(Valence)*

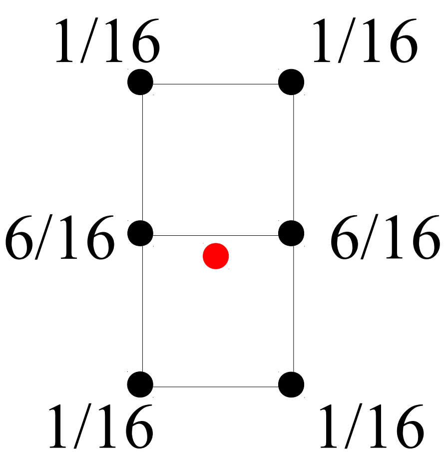


- « face » vertices
- « edge » vertices
- « corner » vertices

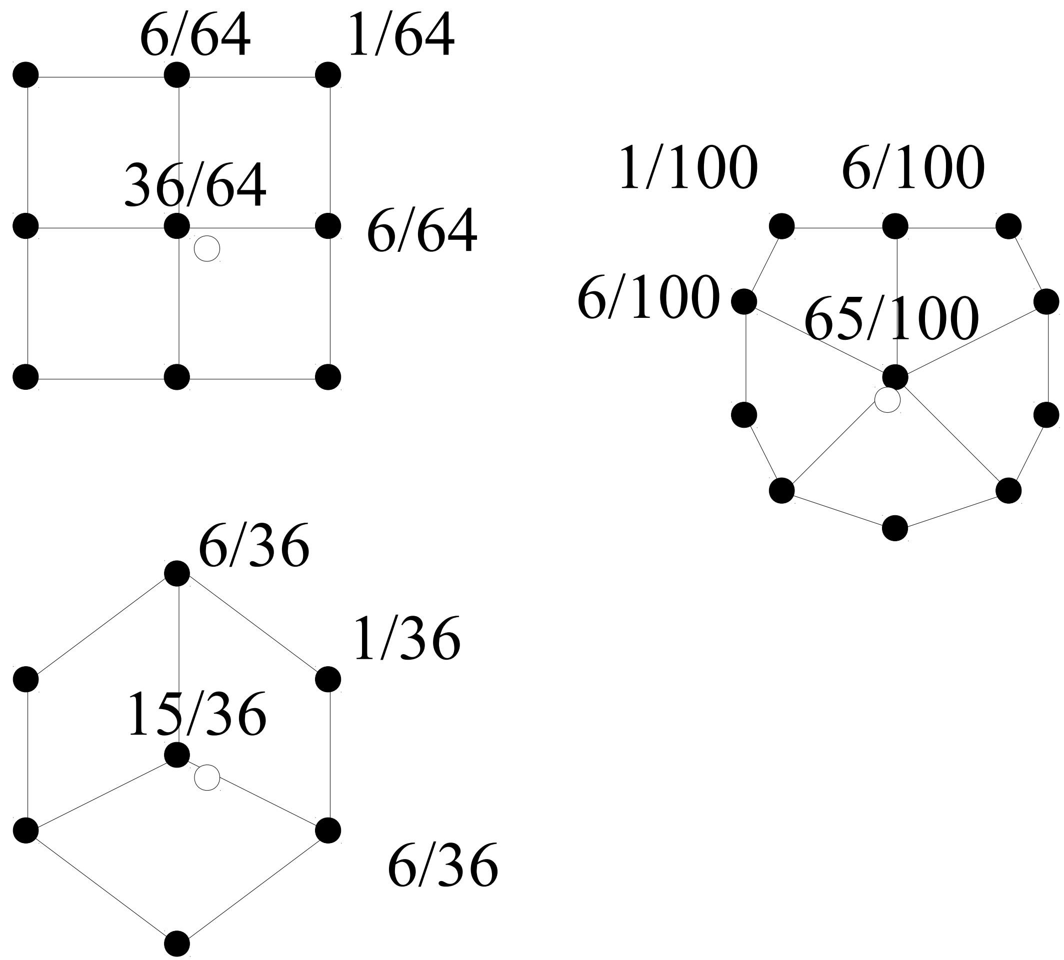
Subdivision surfaces



Face vertex



Edge vertex



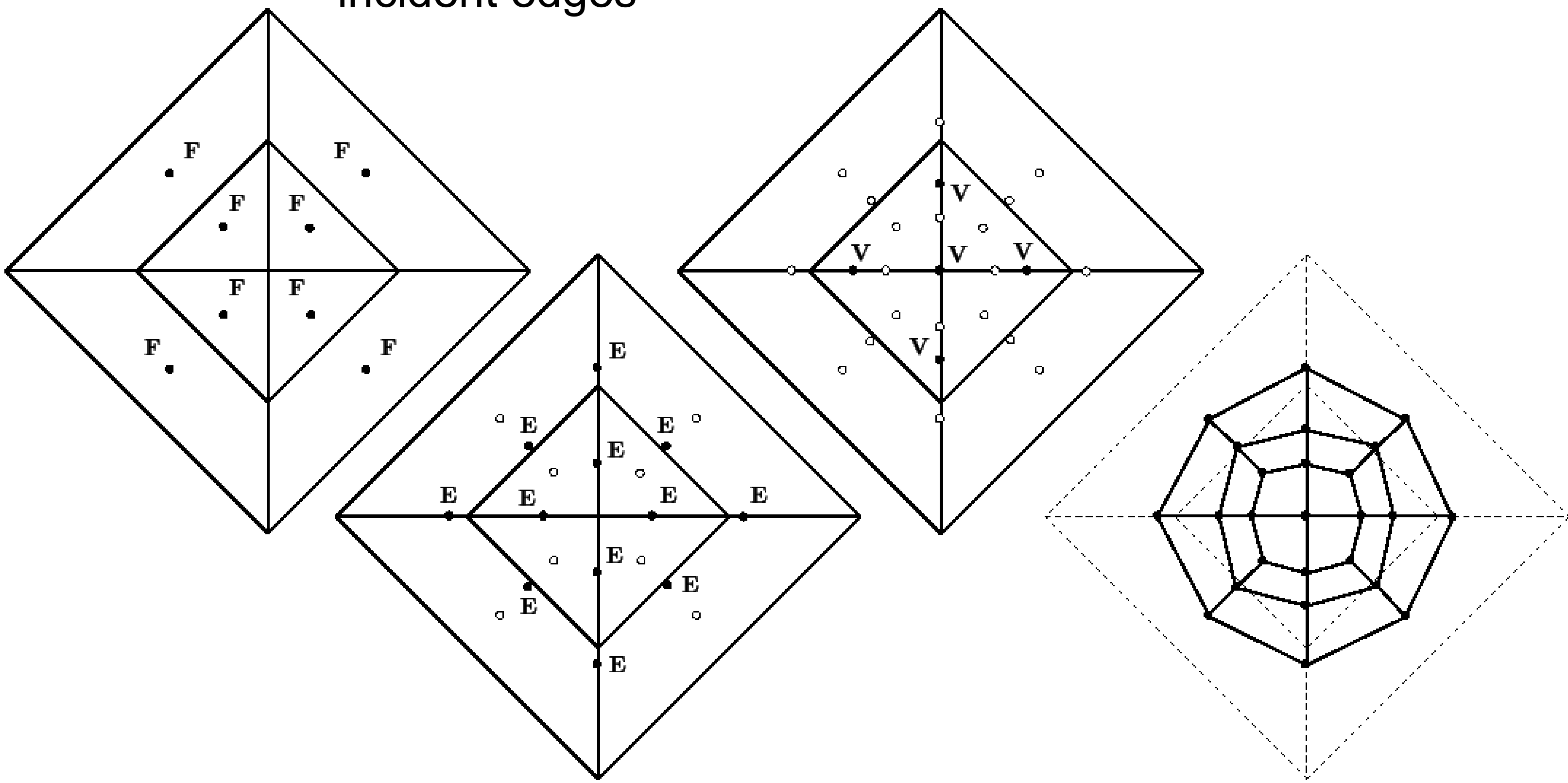
Corner vertex (depends on the valence)

Subdivision surfaces

- Reconnecting the new vertices

  1 – Connect the « face » vertices to the « edge » vertices of neighbouring edges

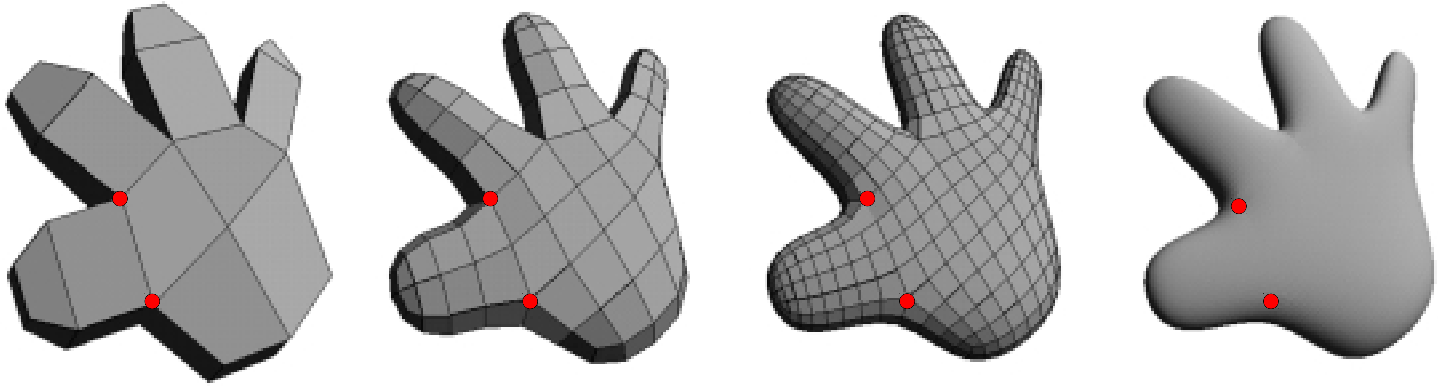
  2 – Connect the « corner » vertices to the « edge » vertices of incident edges



Images : Ken Joy - UCLA

Subdivision surfaces

- As with Doo-Sabin surface, the continuity is degraded for some extraordinary vertices. The bicubic surfaces are therefore C^2 everywhere except at extraordinary points : it is then only C^1.



Images : Imorasa Suzuki

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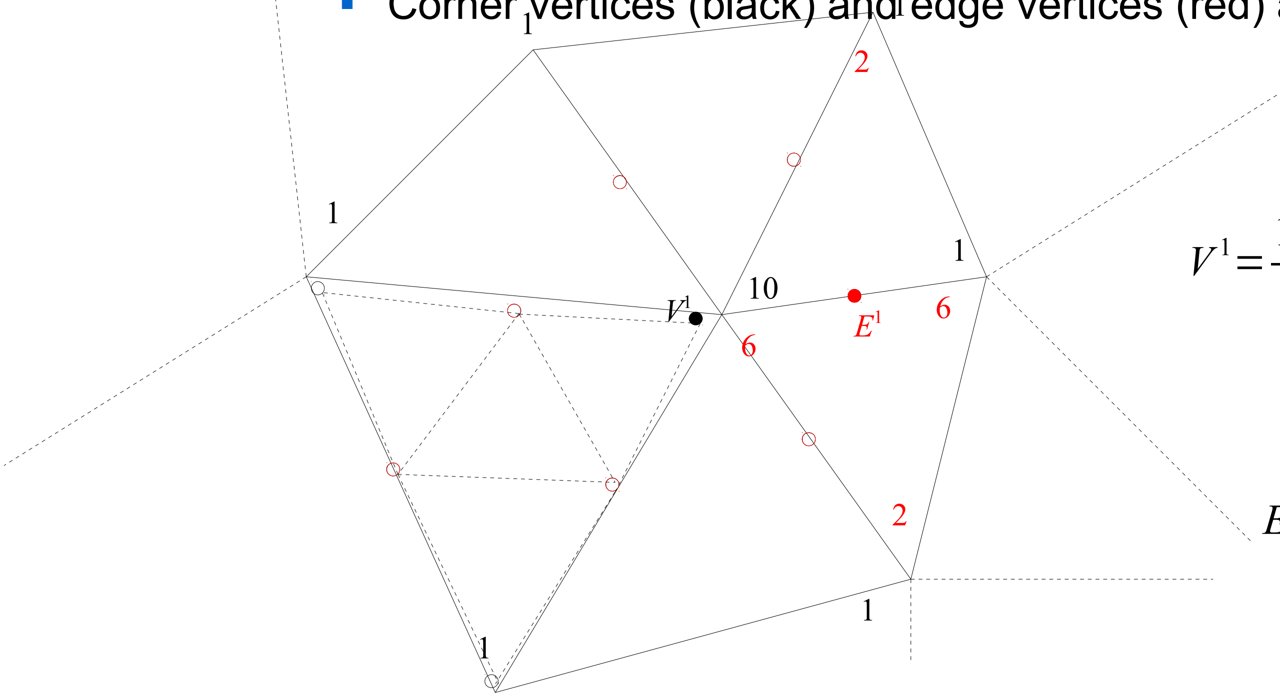
Subdivision surfaces

- Loop's scheme
  - Allows to subdivide triangular meshes
  - The limiting surface is C^2 , except at extraordinary vertices of valence <>6 , where it is only C^1.
  - The principle is to subdivide triangles into 4 sub-triangles.
    - Corner vertices (black) and edge vertices (red) are created.



$$V^1 = \frac{10\,V + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6}{16}$$

$$= \frac{5}{8} V + \frac{3}{8} Q$$

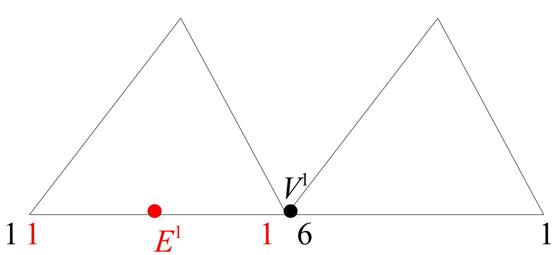
$$E^1 = \frac{6\,V_1 + 6\,V_2 + 2\,F_1 + 2\,F_2}{16}$$

Subdivision surfaces

- As such, works only for vertices with a valence equal to 6
- It may be extended to other valences, but the formula has to be adapted such that the resulting surface is smooth.
- Let $V^1 = \alpha_n V + (1-\alpha_n) Q$

  with $\alpha_n = \left(\frac{3}{8} + \frac{1}{4}\cos\frac{3\pi}{n}\right)^2 + \frac{3}{8}$

  n is the valence of the original vertex.
- On boundaries : vertices should not move inside the surface, they should rather slide along the boundary. One recovers the classical cubic B-Spline scheme in that case



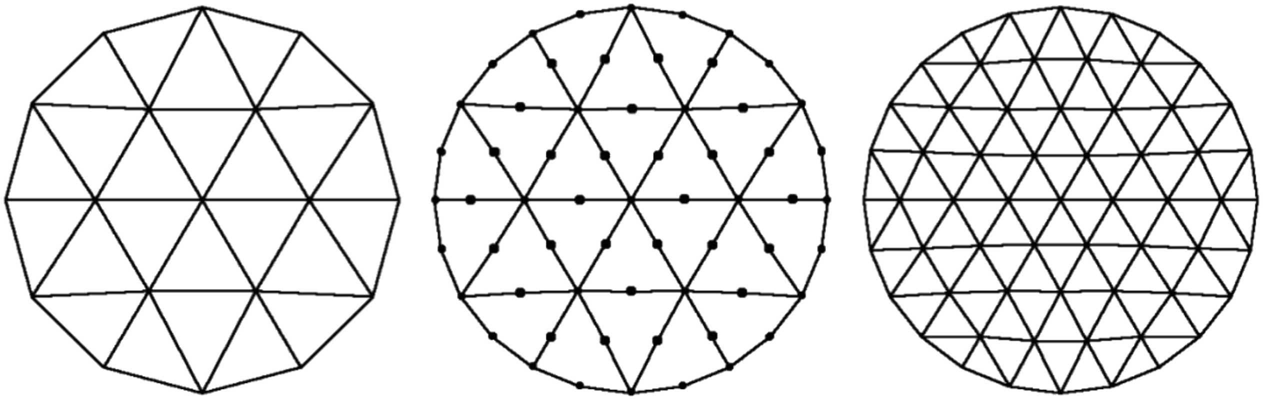
$$E^1 = \frac{V_1 + V_2}{2}$$

$$V^1 = \frac{6}{8} V + \frac{Q_1^* + Q_2^*}{8}$$

(only if the vertices are neighbours on the boundary)

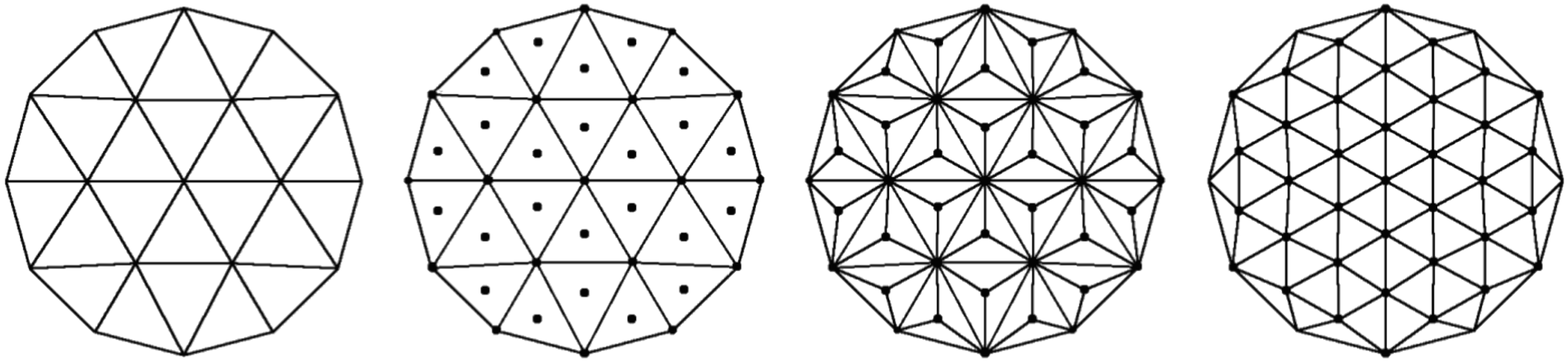
Subdivision surfaces

- Loop's scheme



Subdivision surfaces

- Kobbelt's $\sqrt{3}$ scheme
  - See paper on the course's website
  - For a similar level of refinement, it generates less triangles than Loop's scheme



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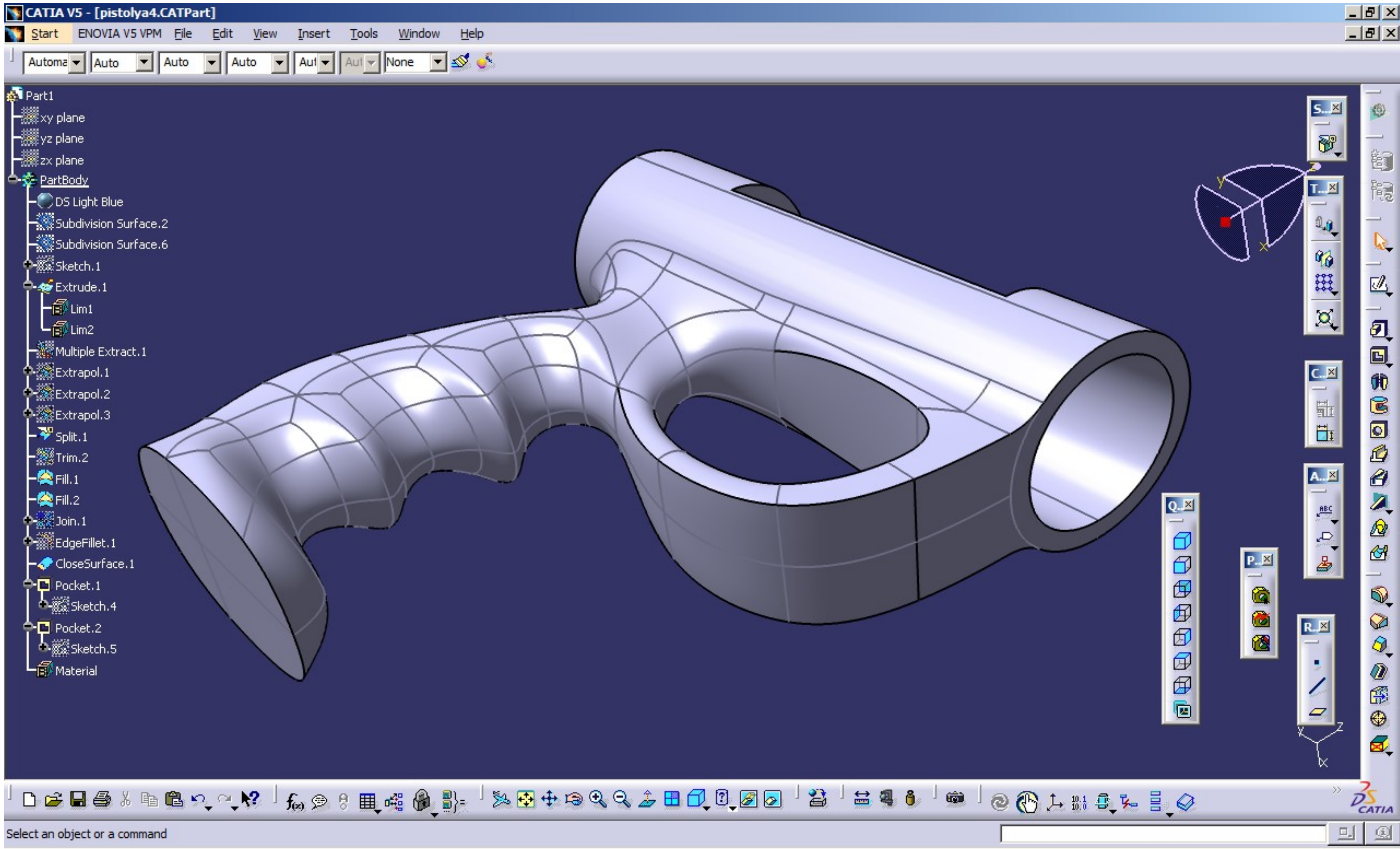
Subdivision surfaces

- Subdivision surfaces in CATIA

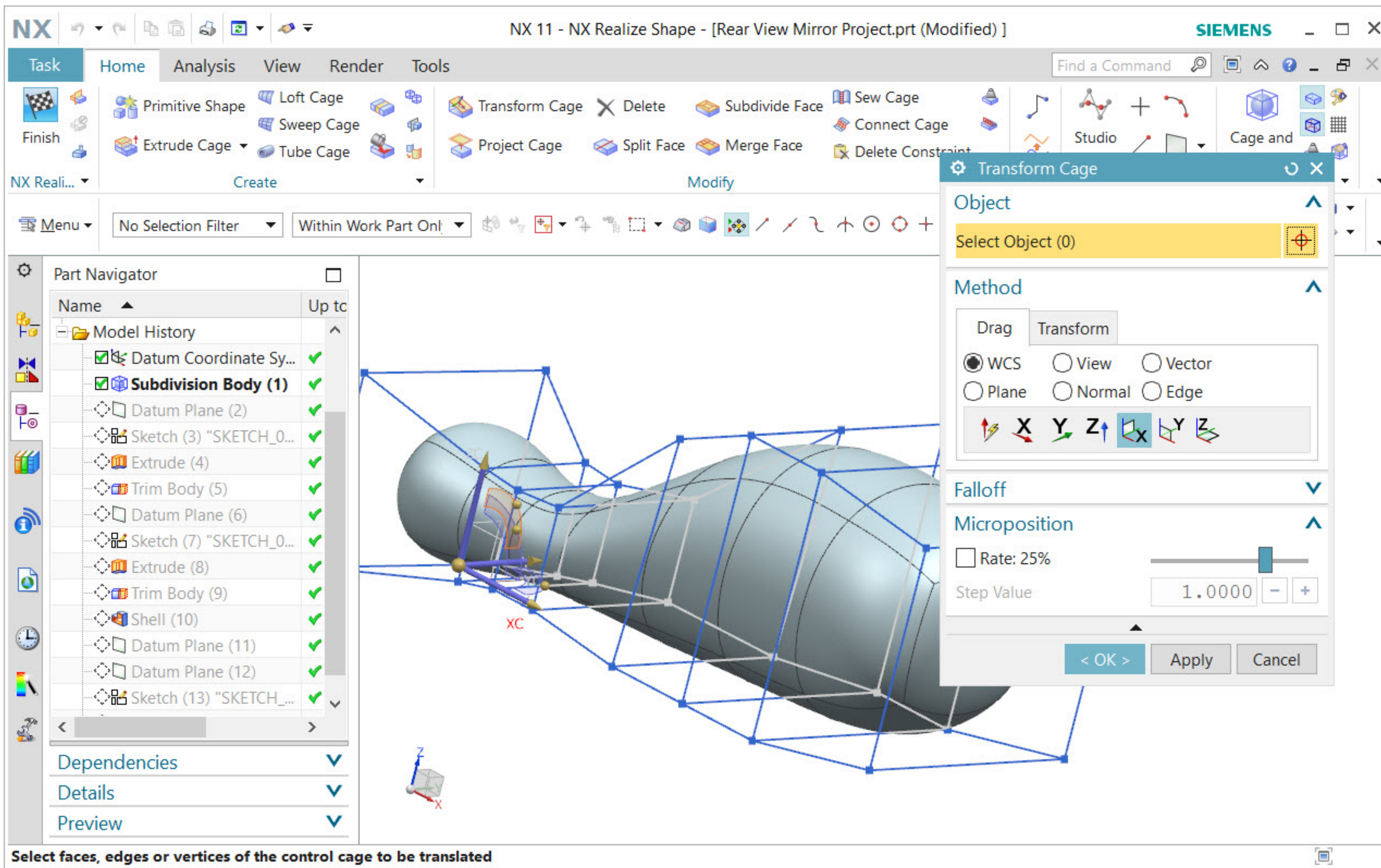
- A very easy-to-use design tool
- As S-s are equivalent to some class of B-Spline surfaces, they retain a good degree of accuracy
- “CATIA Shape” module Imagine Shape (IMA) tool



Subdivision surfaces

- … and in NX

- A very easy-to-use design tool
- As S-s are equivalent to some class of B-Spline surfaces, they retain a good degree of accuracy
- “NX Realize shape” tool



Architectural applications

- Catia in architecture
  - Frank O. Gehry (Fish sculpture , Barcelona ,1992)

Architectural applications

- Catia in architecture
  - Frank O. Gehry (Guggenheim museum, Bilbao ,1997)

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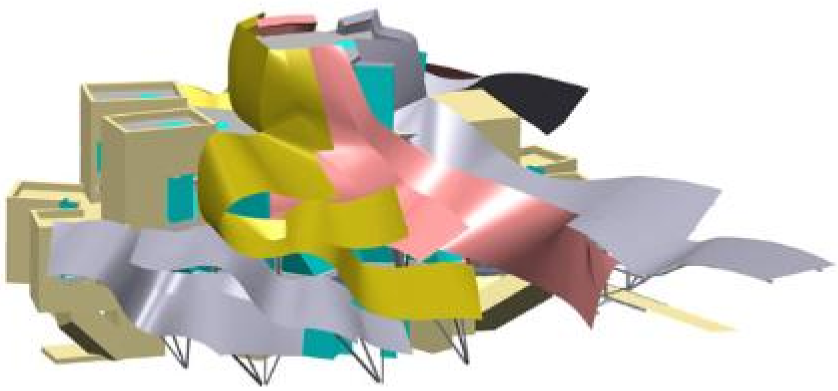
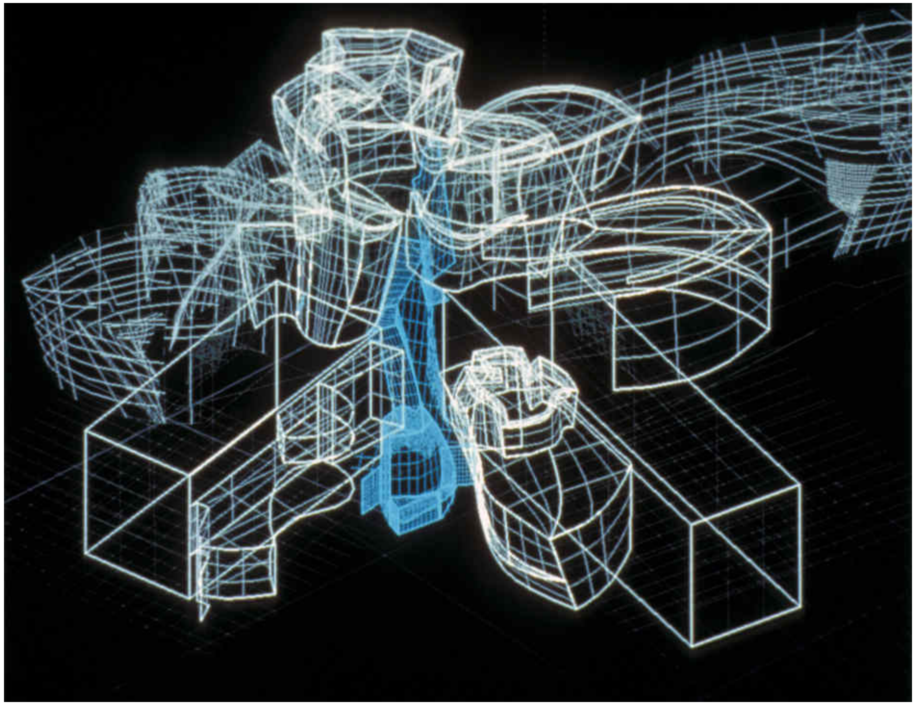
Architectural applications

- Catia in architecture
  - Frank O. Gehry (Guggenheim museum, Bilbao ,1997)





Architectural applications

- Catia in architecture
  - Frank O. Gehry (Walt Disney concert hall, Los Angeles ,2003)

Carol M. Highsmith

Splines of all kinds

- Zoo of Splines ...

Lg-splines
Analytic splines
Parabolic Arc splines
Beta splines
B-splines
Bernoulli splines
Box splines
Cardinal splines
Circular splines
Complete splines
Nu-splines
Natural splines
L-splines
Whittaker splines

Nonlinear splines
D^m splines
Discrete splines
Euler splines
Exponential splines
Gamma splines
GB-splines
HB-splines
Hyperbolic Splines
Complete Monosplines
Tchebycheffian splines
Tension splines
Trigonometric splines
Bézier splines

One-sided splines
Parabolic splines
Perfect splines
Periodic splines
Poly-splines
Rational splines
Simplex splines
Spherical splines
Taut splines
Complex splines
Confined splines
Deficient splines
Thin plate splines