CAD & Computational Geometry

Course plan

- Introduction
- Segment-Segment intersections
- Polygon Triangulation
- Intro to Voronoï Diagrams & Delaunay Triangulations
- Geometric Search
- Sweeping algorithm for Voronoï Diagrams

CAD & Computational Geometry

Voronoi Diagrams

- Voronoi Diagrams or Dirichlet Tessellations
  - In dimension 2 or 3

    Johann Peter Gustav Lejeune Dirichlet (1805-1859) ~ 1850.
  - For all dimensions :

    Georgy Feodosevich Voronoy (1868-1908) ~ 1908

G.F. Voronoï (1908). "Nouvelles applications des paramètres continus à la théorie des formes quadratiques". Journal für die reine und angewandte Mathematik 134: 198–287.

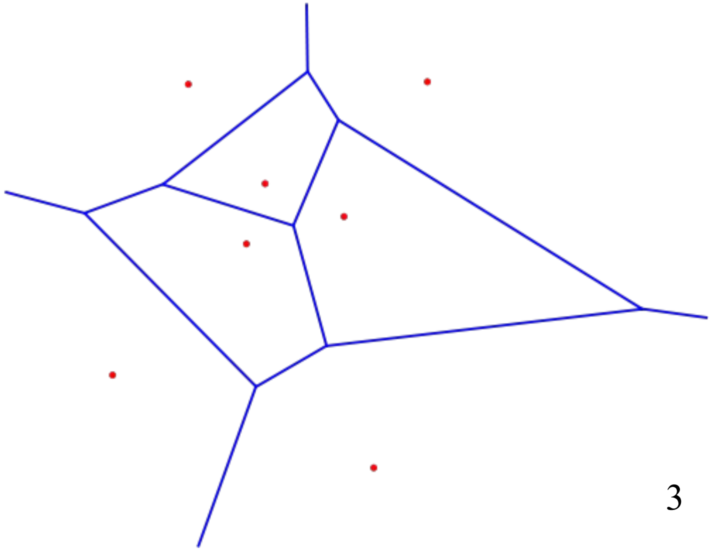
Voronoi Diagrams

- Let $P=\{ p_0, p_1, \ldots , p_{n\text{-}1} \}$ a point set in the plane
- The Voronoi diagram $Vor(P)$ is a paving of the plane into n cells $V(p_i)$.
- Inside cell $V(p_i)$, the closest point belonging to P is p_i.
  - Here, the we consider the euclidean distance :

    $$Dist(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$$

  - A point q is inside the cell corresponding to point p_i iff

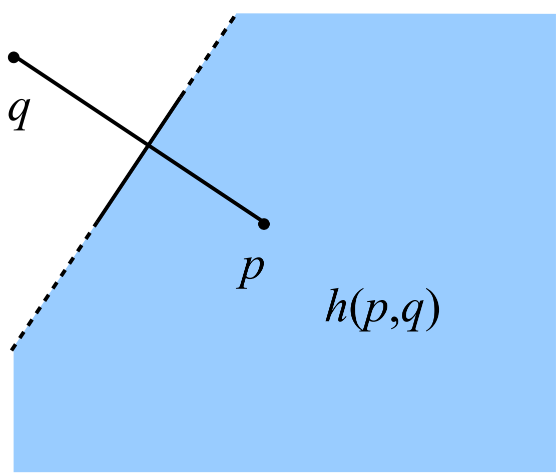
    $$Dist(q, p_i) < Dist(q, p_j) \quad \forall p_j \in P, j \neq i$$

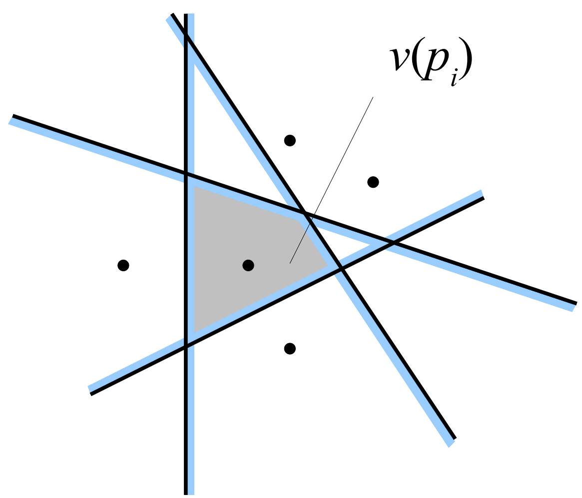


Voronoi Diagrams

- What is the structure of the Voronoi diagram ?
  - The bissector of points p and q separates the plane in two half-planes
  - Let $h(p,q)$ the open half plane containing p
  - We have then $r \in h(p,q)$ iff $Dist(r,p) < Dist(r,q)$

    and $V(p_i) = \cap_{j \neq i} h(p_i, p_j)$
    - $V(p_i)$ is the intersection of the n-1 half-planes, a possibly open polygonal region, limited by at most n-1 edges and n-1 vertices. The shape is necessarily convex.





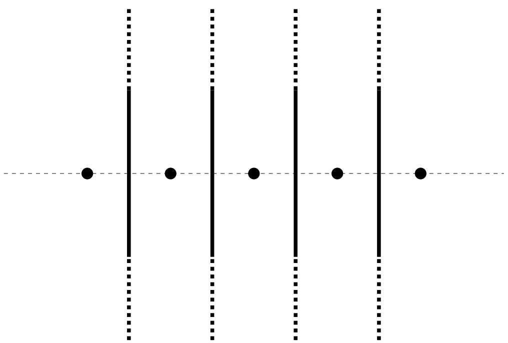
- $Vor(P)$ is either

  a) a set formed by n-1 parallel lines

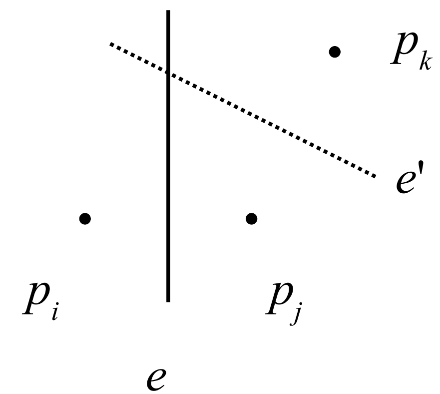
  b) a connected set of line segments or half-lines.

Voronoi Diagrams

- Demonstration
  - Case a) straightforward
  - Case b) : Let us consider that an edge e that bounds two Voronoi cells $V(p_i)$ and $V(p_j)$ is a line : it separates the plane in two parts. Let p_k a third point, not colinear with $p_i p_j$: necessarily, the edge e' bounding $V(p_j)$ and $V(p_k)$ is not parallel to e and intersects it. The part of e which is in $h(p_k,p_j)$ cannot be on the boundary of $V(p_j)$ since it is closer to p_k than p_j.
  - Therefore, the edges of the Voronoi diagram are either half-lines or line segments, or only complete lines as in the case a)



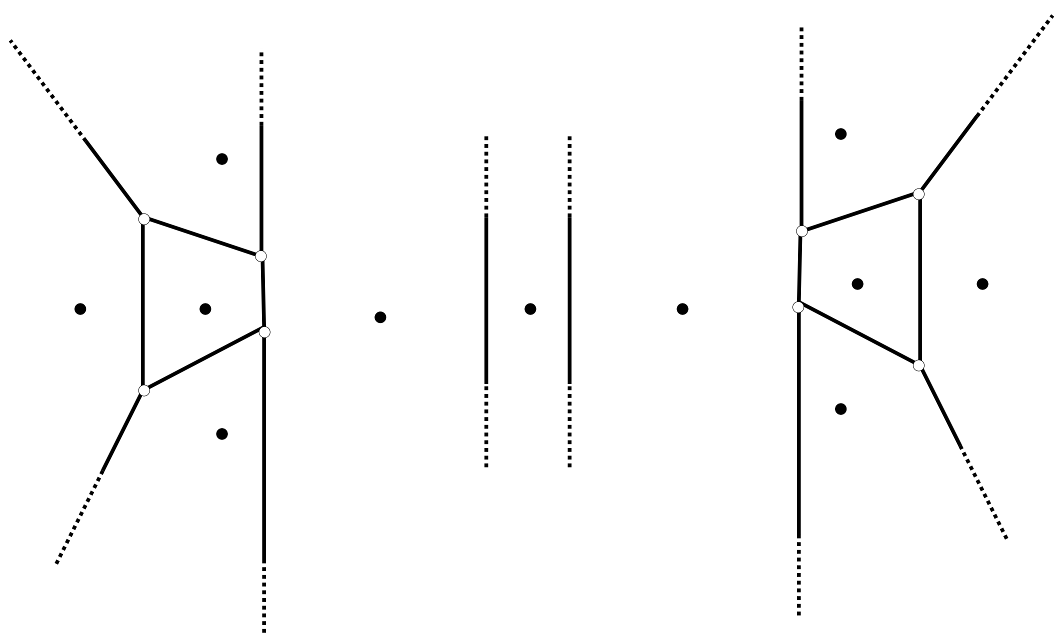
n-1 parallel lines



Voronoi Diagrams

- There remains to prove that the Voronoi diagram is a connected graph.

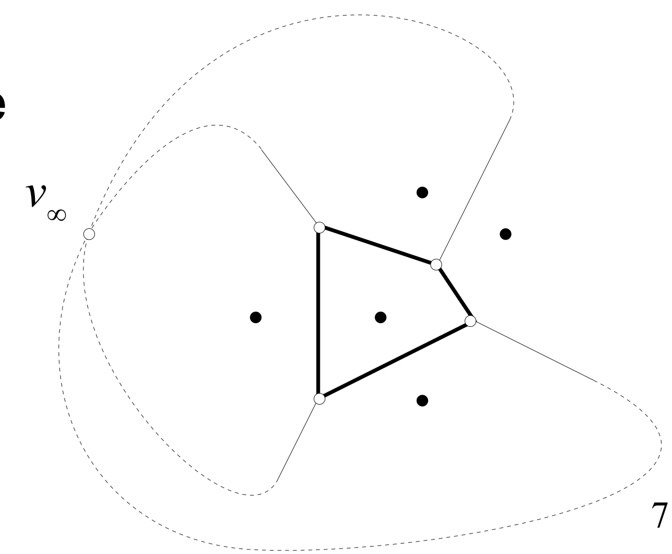
  Let us suppose that it is not the case. At least one Voronoi cell woud separate the plane in two parts. Since Voronoi cells are convex, its bundaries are necessarily made of at least some complete lines, which is a contradiction with the previous demonstration (either only complete lines; or only line segments and half-lines).



- Complexity of the Voronoi diagram

  What is the number of cells, edges and vertices ?

  - There are n sites in P ; and in the worst case, a cell may have n-1 edges and vertices. The global complexity of $Vor(P)$ *could* be quadratic. In fact, it is not the case:
  - Euler formula for a plane graph (including the unbounded face around the graph) : $v - e + f = 2$
    Here, we consider an additional vertex infinitely far away ; we have therefore
    $n_v + 1 - n_e + n = 2 \rightarrow 2n_v - 2n_e + 2n = 2$
  - Each edge has two vertices ; the sum of the valence of each vertex is twice the number of edges. Also, the valence is at least 3.



$$2n_e \geq 3(n_v + 1)$$

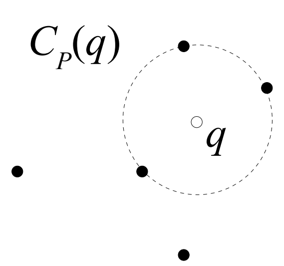
$$2n_v - 3(n_v + 1) + 2n \geq 2 \rightarrow n_v \leq 2n - 5$$

$$2n - 5 + 1 - n_e + n \geq 2 \rightarrow n_e \leq 3n - 6$$

- The number of bisectors is quadratic, but only a small fraction (linear in n) are part of the Voronoi diagram.
  - Which ones ?

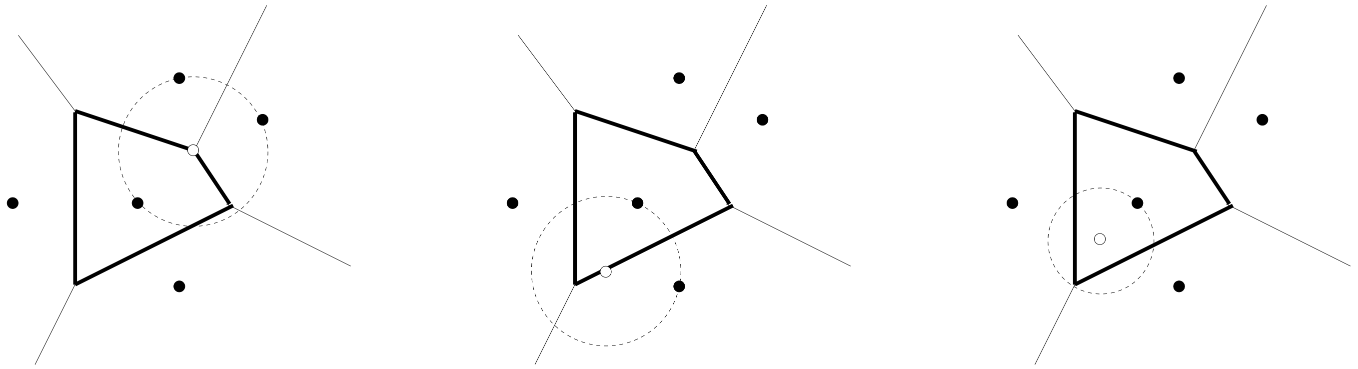
    One defines $C_P(q)$ as the greatest circle of center q which does not contain any point p_i of P in its interior. It has at least one point of P on its boundary.

    Then :
    - One point q is a vertex of $Vor(P)$ *iff* $C_P(q)$ has at least three points of P on its boundary.
    - A bissector of points p_i et p_j exists as an edge in $Vor(P)$ iff there are points q on it such that $C_P(q)$ has p_i and p_j on its boundary but not any other point of P.



Voronoi Diagrams

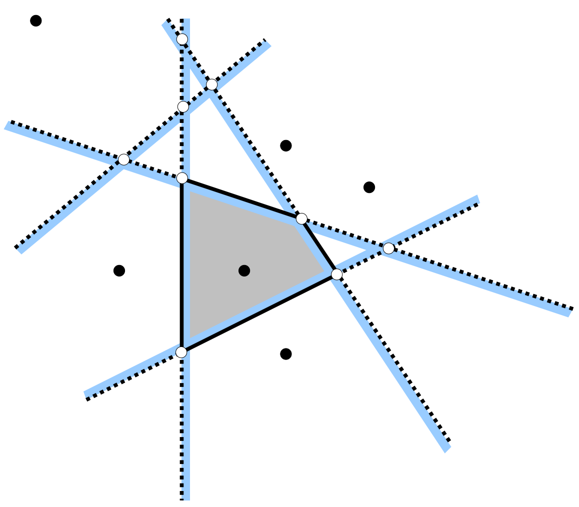
- Demonstration
  - 3 points or more on the boundary → Center of the circle is a vertex of the Voronoi diagram
  - 2 points on the boundary → Center is on an edge of the Voronoi diagram
  - Only one point on the boundary → Center located inside a cell of the Voronoi diagram



Voronoi Diagrams

- How to compute the Voronoi diagram
  - For each cell $V(p_i)$, compute the intersections of all the half-planes $h(p_i,p_j)$ with $j \neq i$ using the intersection of lines of chapter CG2

    Complexity : in $n \log n$ for each cell (one does not know in advance which intersections will be found in the final shape of the cell)
  - There are n cells → $n^2 \log n$ globally. However, the complexity of the diagram is only $O(n)$...
  - Is it possible to be faster ? → yes !
    - Optimum : $\Omega(n \log n)$
    - Demonstration later on.



Delaunay Triangulations

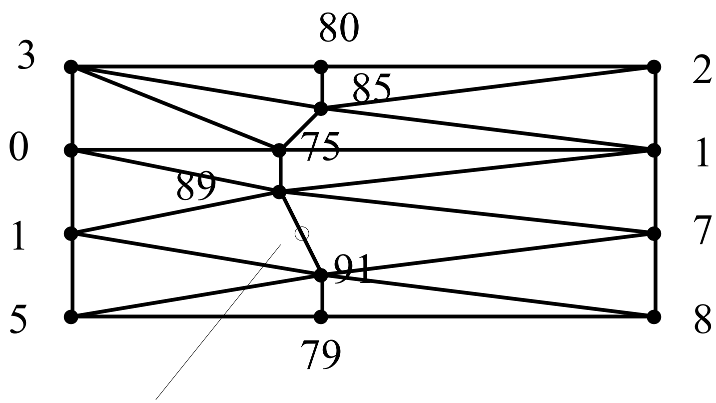
Delaunay Triangulations

- Boris Nicolaïevich Delaunay (Delone) (1890 - 1980)
  - Was a student of G. Voronoi
  - Developed a theoretical study of triangulation that bear his name in the 30's

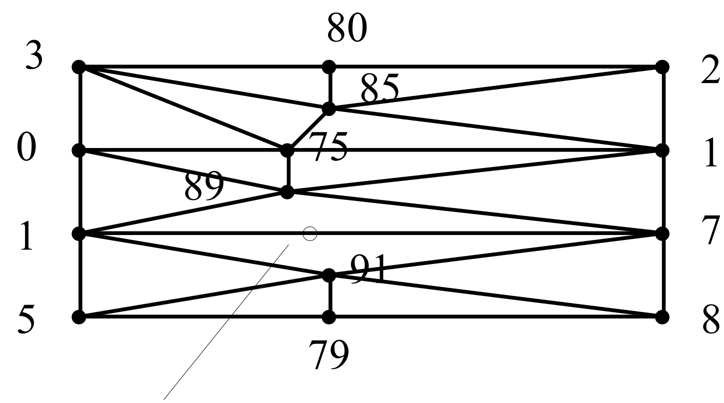
B. Delaunay: Sur la sphère vide. A la mémoire de George Voronoi,
Известия Академии наук СССР.
Отделение математических и естественных наук 6:793–800, 1934
Bulletin de l'Académie des sciences de l'URSS.
Classe des sciences mathématiques et naturelles , 6:793–800, 1934

Delaunay Triangulations

- Interpolation and why certain triangulations are “better” than others



90 (seems legit)



3 (seems wrong)

  - On the right, the interpolation is using samples that geometrically far apart
  - There exist “better” triangulation than others with respect to interpolation. The Delaunay triangulation is one triangulation among others, it has interesting properties precisely if the triangulation is used as a basis for interpolation

Delaunay Triangulations

- Some characteristic properties of a triangulation
  - Let P a set of vertices $\{p_0 \dots p_{n\text{-}1}\}$, not all collinear, i.e. in general position
  - Let S be a maximal subdivision of the plane, such that no new edge can join two existing vertices without cutting an existing edge in S.
  - Necessarily, the edges of the convex hull belong to S.
  - A triangulation of P is precisely such a subdivision S for which the vertices of the subdivision are taken in P.
  - The cells are triangles, as we have seen that any polygon can be triangulated with a constant number of triangles.
  - What is the complexity of a triangulation (any triangulation) ?

    Let k be the number of vertices of P located on the convex hull of P. Then, any triangulation of P has $2n–k–2$ triangles, and $3n–k–3$ edges.

Delaunay Triangulations

- Some characteristic properties of a triangulation

  Let k be the number of vertices of P located on the convex hull of P. Then, any triangulation of P has $2n–k–2$ triangles, and $3n–k–3$ edges.

  - Proof : Let m be the number of triangles. In the planar graph, there are therefore $n_f=m+1$ faces counting the external boundless face.
    Every triangle has three edges, the boundless face has k edges.
    Every edge is incident to exactly two faces

    The total number of edges is then $n_e=(3m+k)/2$

    Euler's formula must hold for a planar graph, so

    $n–n_e+n_f= 2$

    Replacing n_e,and n_f one gets $m=2n–k–2$ and therefore $n_e=3n–k–3$.

  - These bounds are valid in 2D. En 3D, these are quadratic (as seen earlier)

Delaunay Triangulations

- Some characteristic properties of a triangulation
  - Let T be the triangulation, and suppose it contains m triangles

    Let us consider the $3m$ angles found at vertices of the triangles of T , sorted by increasing order.

    Let $\alpha_0,\alpha_1,\ldots,\alpha_{3m-1}$ be the sequence of such angles. Here, $\alpha_i \leq \alpha_j$ for all $i < j$.

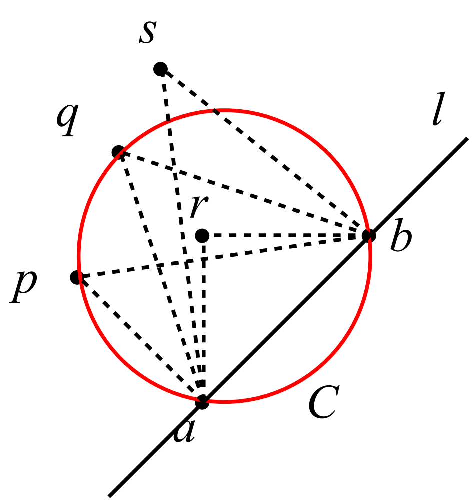
    $A(T)$ is the "angular vector" in this ordering.
  - Let T' be another triangulation of P (with **as many** triangles), and let $A(T')$ be its angular vector.

    Let write $A(T) > A(T')$ (lexicographically greater) iff :

    $\alpha_j = \alpha_j'$ for all $j < i < 3m$, and $\alpha_i > \alpha_i'$.
  - A triangulation T is said to be angularly optimal if for any other triangulation T' one has $A(T) \geq A(T')$

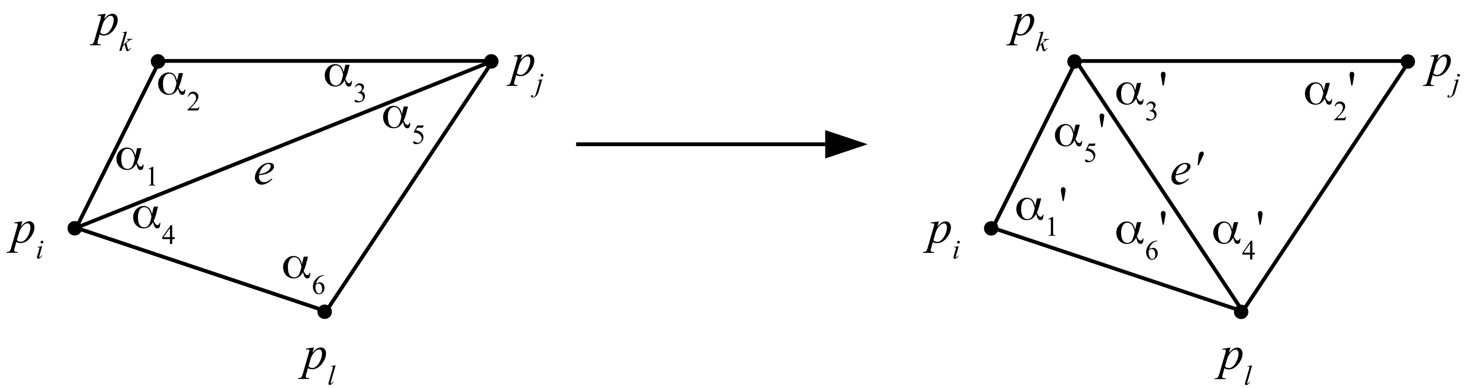
- Thales' theorem (the other one ...)
  - Let l be a line intersecting a circle at two points points a and b. Let r be a point strictly inside the circle, and p and q two points on the circle, and finally s a point strictly outside the circle

    Then, $\widehat{arb} > \widehat{apb} = \widehat{aqb} > \widehat{asb}$
  - Proof : by constructing the locus of the points q such that $\widehat{aqb} = const$ (a circle !)



Delaunay Triangulations

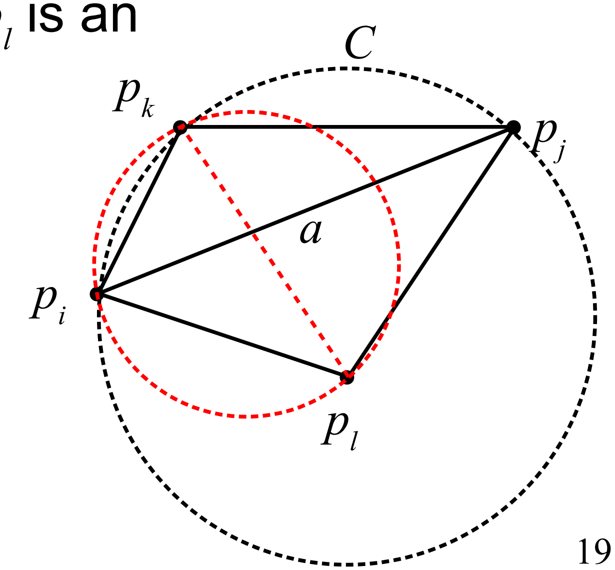
- Edge swapping
  - Let us consider an edge $e = p_i p_j$ of a triangulation T from P. If this edge is internal, then it has two neighbors : the triangles $p_i p_j p_k$ and $p_j p_i p_l$. If these form a convex polygon one can get a different triangulation T' by swapping the edge e.
  - The only difference in the angular vector is that the six six angles $\alpha_0 .... \alpha_6$ of $A(T)$ are replaced by $\alpha_0' .... \alpha_6'$ of $A(T')$.



  - Let us call $e = p_i p_j$ an **illegal** edge if $\min_{i \le i \le 6} \alpha_i \ < \ \min_{i \le i \le 6} \alpha_i'$
  - In other words, an edge is illegal if one can make the minimal angle greater by performing an edge swap on this edge ans make it **legal**

Delaunay Triangulations

- Edge swapping
  - Let T be a triangulation with an illegal edge e. Let T' the triangulation obtained by swapping the edge e. Necessarily $A(T') \geq A(T)$.
  - In fact, it is not necessary to compute all 12 angles to check that an edge is illegal, thanks to the second theorem of Thales. Let $p_i p_j$ be an edge incident to triangles $p_i p_j p_k$ and $p_i p_j p_l$, and C the circle going through $p_i p_j p_k$. **The edge $p_i p_j$ is illegal if p_l is inside the circle C.** Moreover, if p_i, p_j , p_k and p_l form a convex polygon, and are not all on the same circle, then only one of $p_i p_j$ or $p_k p_l$ is an illegal edge. The other is necessary legal.



Delaunay Triangulations

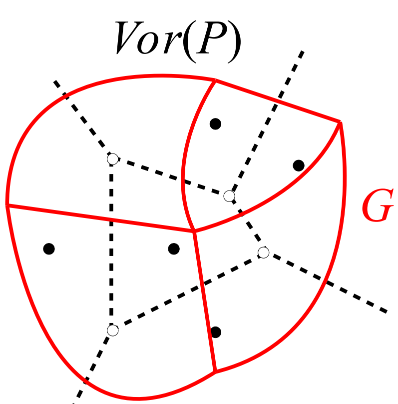
- Legal triangulation
  - It is a triangulation for which all the edges are legal.
  - The algorithm to get such a triangulation from any other valid triangulation is as follows :

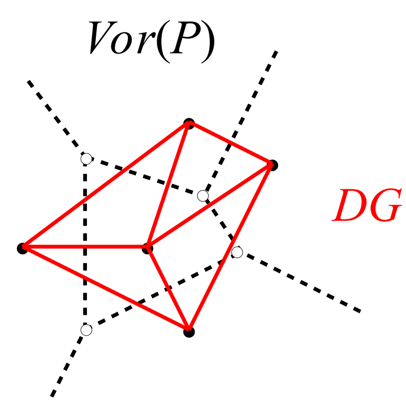
LegalTriangulation(*T*)
Input : an arbitrary triangulation T of the set P
Output : a legal triangulation of P
{
While T contains an illegal edge $p_i p_j$
{
Let $p_i p_j p_k$ and $p_j p_i p_l$ the two adjacent triangles to $p_i p_j$
erase $p_i p_j$ and add $p_k p_l$
}
Return T.
}

  - This triangulation always exists because it is always possible to swap an illegal edge (and make $A(T)$ better). Each swap is making $A(T)$ better; therefore, one does never swap the same edge twice (there are no cycles) and the algorithm always finishes as the number of triangulation for a given set of points, although very high, is finite.

- Delaunay Graph
  - Let G the dual graph of the Voronoï diagram $Vor(P)$

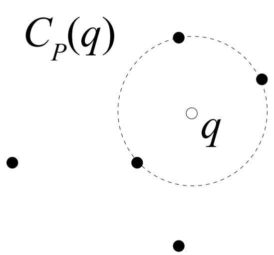
    The vertices of G are the cells of $Vor(P)$; the cells of G are the vertices of $Vor(P)$. An arc links two vertices of G iff two cells of $Vor(P)$ share an edge.
  - Let DG the Delaunay graph, simply G for which the vertices are those of the set P. The edges of this graph are straight lines linking those vertices.





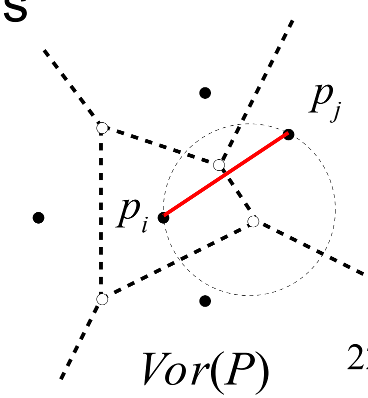
- Properties of the Delaunay graph
  - If P is a set of points in the plane, it is necessarily a planar graph – i.e. no two edges are intersecting.

  Proof : from the properties of the vertices and edges of $Vor(P)$:
    - A bissector between vertices p_i and p_j exists as an edge in $Vor(P)$ iff there is a point q on the edge such that $C_P(q)$ has p_i et p_j on its boundary, and no other vertex of P.



  In the case of the Delaunay graph, it can be rewritten:
    - An edge p_ip_j belongs to $DG(P)$ iff there exists a circle going through p_i and p_j and which does not contain any other vertex p_k of P.

  It is therefore clear that the two edges of $DG(P)$ cannot intersect , otherwise it would mean that at least one of the vertices (out of 4) would be inside any circle going through p_ip_j.



- Properties of the Delaunay graph

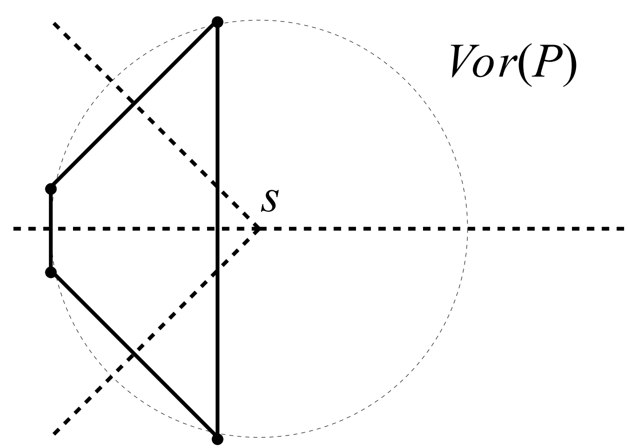
  Reformulating properties of $Vor(P)$

  - Three points p_i, p_j, p_k of P are vertices of the same cell of $DG(P)$ if the circle going through p_i, p_j, p_k does not contain any other point of P
  - Two points p_i, p_j of P define an edge of $DG(P)$ iff there exists at least one circle going through p_i and p_j such that it does not contain any other point of P

  This implies the **Delaunay criterion** (subjected to the generality position of the points p_i) :

  - T is a Delaunay triangulation of P iff the circumscribed circle of any triangle of T does not contain any other point of P in its interior.

- Properties of the Delaunay graph
  - Question related to the non general configurations of the p_i (i.e. k>3 points may be cocyclic)
    - A vertex of the DDV is of valence k
    - The Delaunay graph therefore contains cells with k edges (convex cells).
    - On may triangulate these cells arbitrarily since all vertices are cocyclic, every triangle will respect the Delaunay criterion at the limit (some vertices not belonging to the triangle are exactly on the circumscribed circle...)



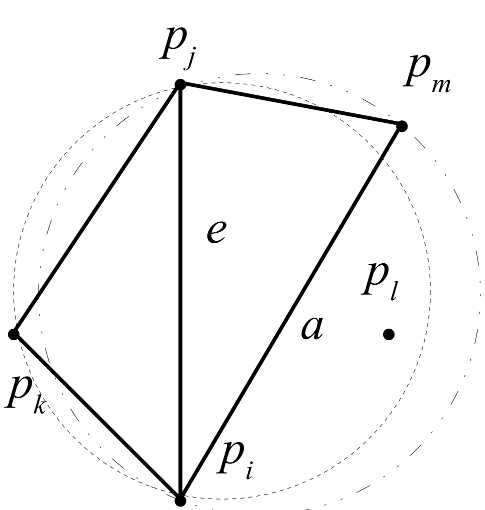
    - Eventually, a Delaunay triangulation is any triangulation satisfying the Delaunay criterion (also called criterion of the empty sphere)- it is not necessarily unique if the vertices are not in general position. It is otherwise unique.

Delaunay Triangulations

- Properties of the Delaunay graph
  - A triangulation T of P is legal only if it is a Delaunay triangulation
  - Proof :It is clear that any Delaunay triangulation is legal (cf p 20). But is every legal triangulation also a Delaunay triangulation ?
    - Yes - proof by contradiction

      Let us suppose that T is a legal triangulation, but not a Delaunay one. Hence, it contains at least a triangle $p_i p_j p_k$ whose circumscribed circle contains at least one other vertex p_l of P. Let us take such a triangle and such a point that, among all of them, maximizes the angle $\widehat{p_i p_l p_j}$

      Let $p_i p_j$ be the edge e such that $p_l p_j p_i$ does not intersect $p_i p_j p_k$, it does belong to T, hence it is legal. Let $p_i p_j p_m$ be the opposite triangle. p_m is not in the circumscribed circle of $p_i p_j p_k$, but the circumscribed circle of $p_m p_j p_i$ contains p_l because it contains an arc of the circumscribed circle of $p_i p_j p_k$ that is on the same side of e as p_l. Let $p_i p_m$ be the edge a such that $p_m p_j p_i$ does not intersect $p_l p_m p_i$. Thus, by the second theorem of Thales, $\widehat{p_i p_l p_m} > \widehat{p_i p_l p_j}$. This contradicts the choice of p_l, therefore p_l does not exist : the triangulation is a Delaunay triangulation.

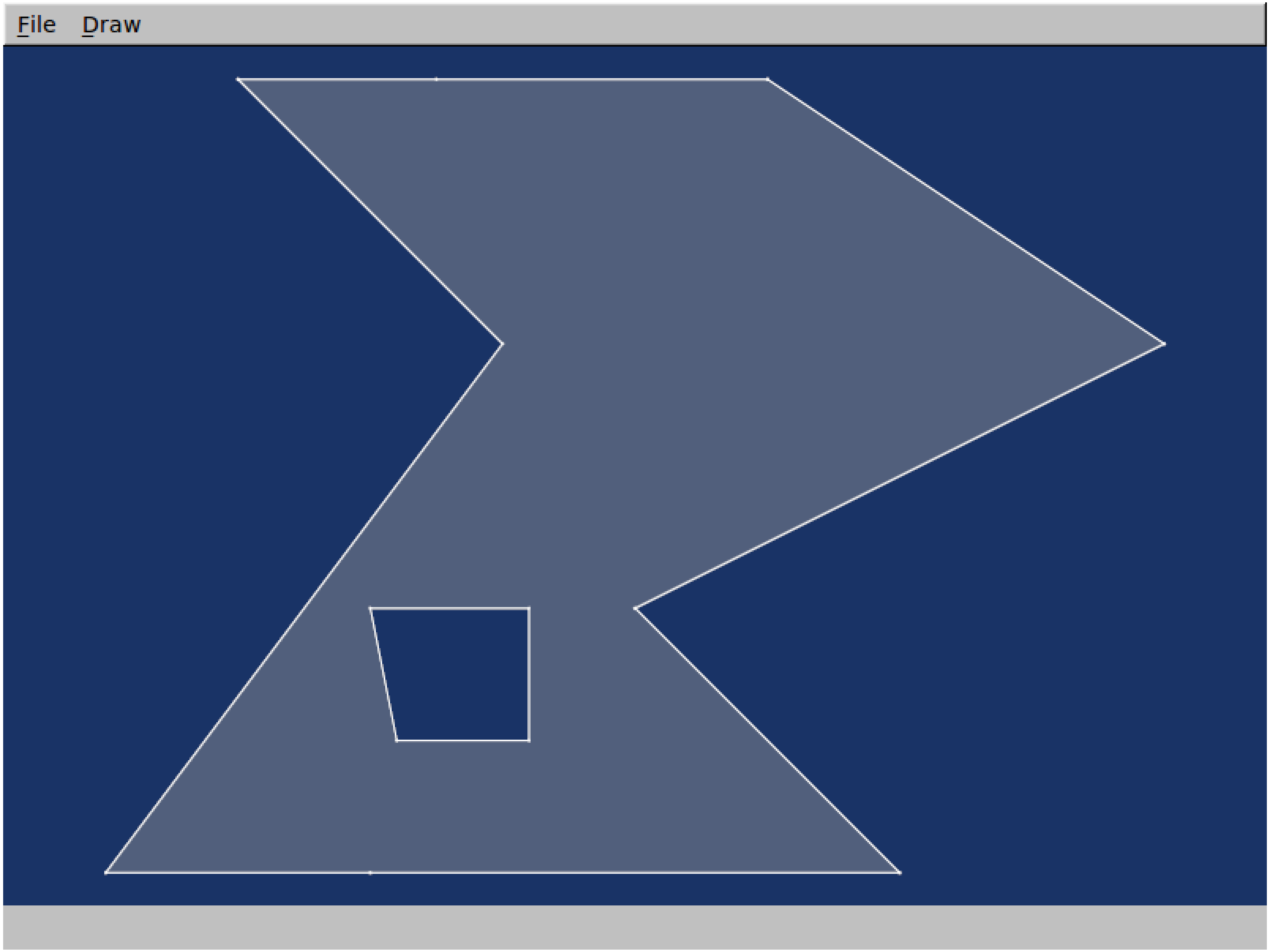


Delaunay Triangulations

- Properties of the Delaunay graph
  - A Delaunay triangulation of P is angularly optimal, i.e. it maximizes the minimal angle of all triangulations of P .
  - Every angularly optimal triangulation of P is a Delaunay triangulation of P, hence the circumscribed circle of every triangle is empty.

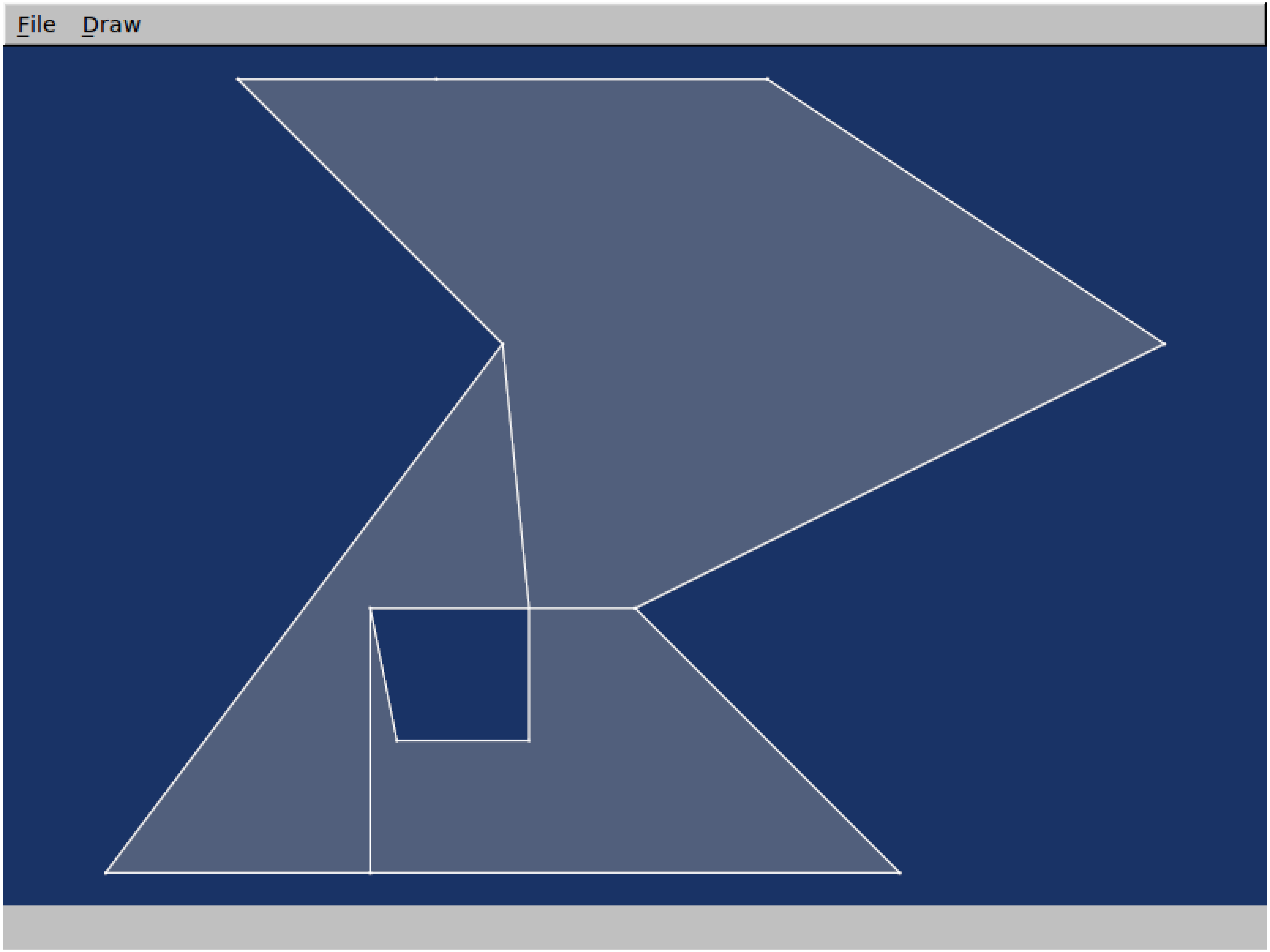
Delaunay Triangulations

- Example : Simple contour



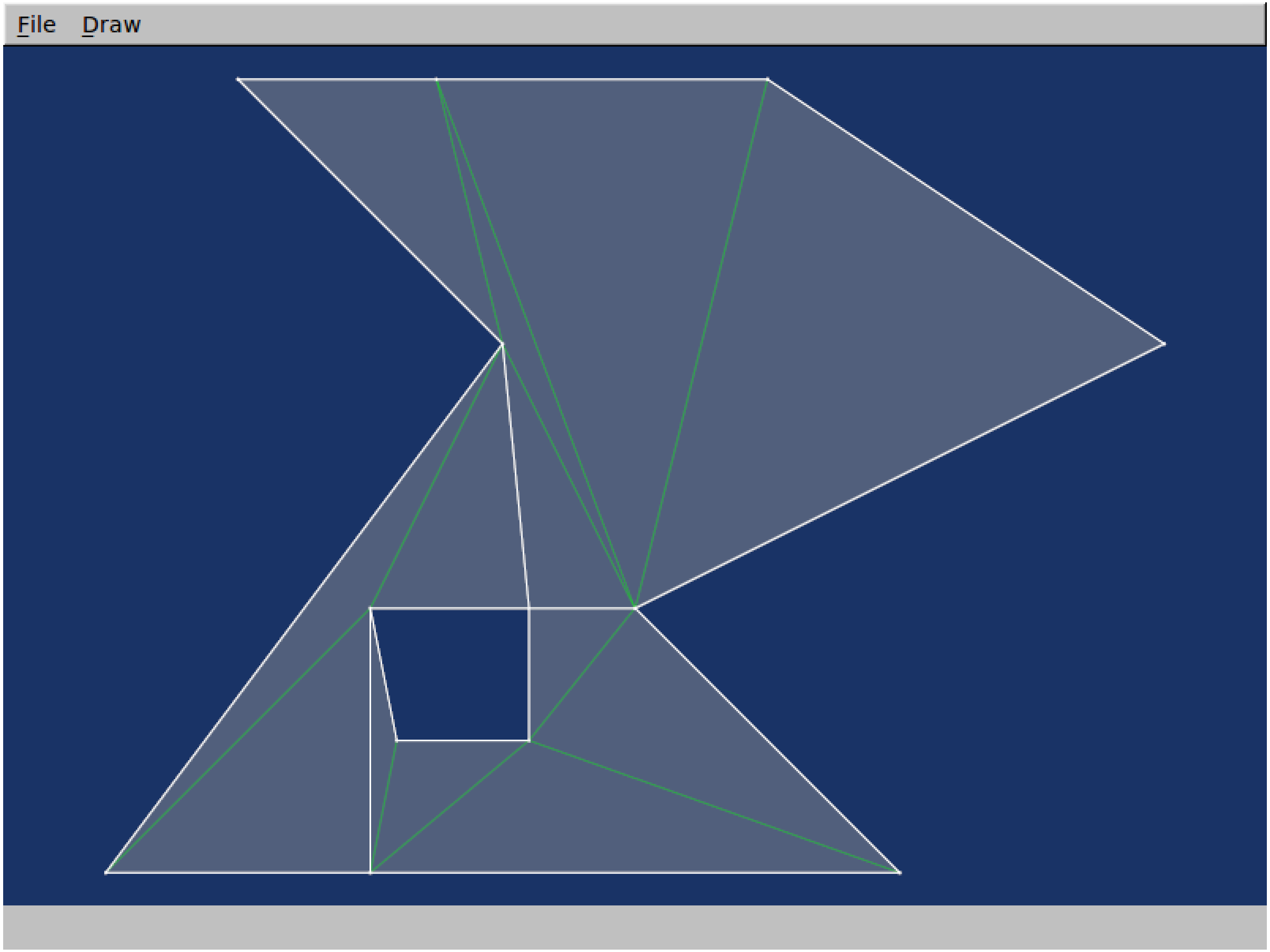
Delaunay Triangulations

- Example : Simple contour – monotone polygons



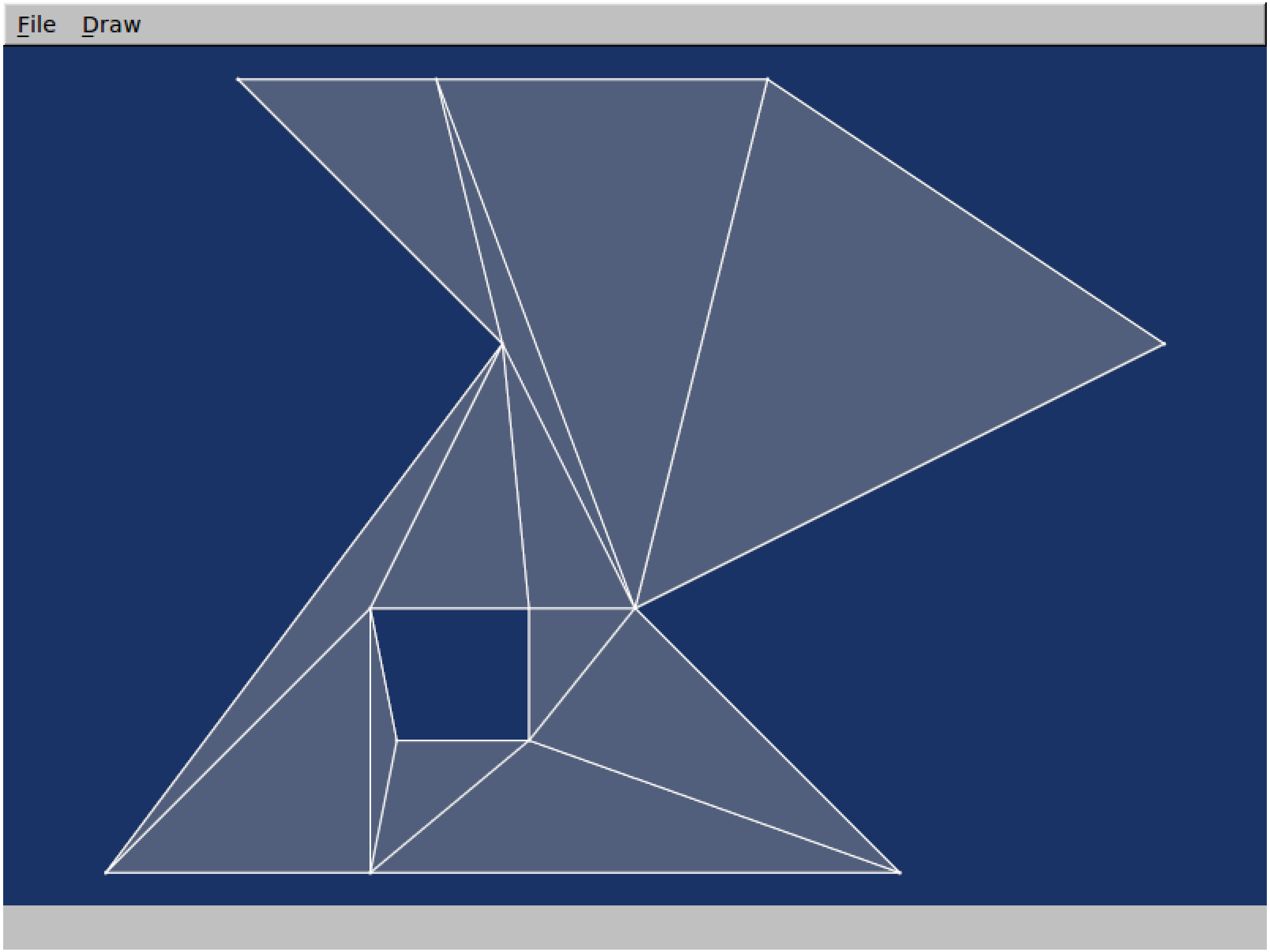
Delaunay Triangulations

- Example : Simple contour – triangulation



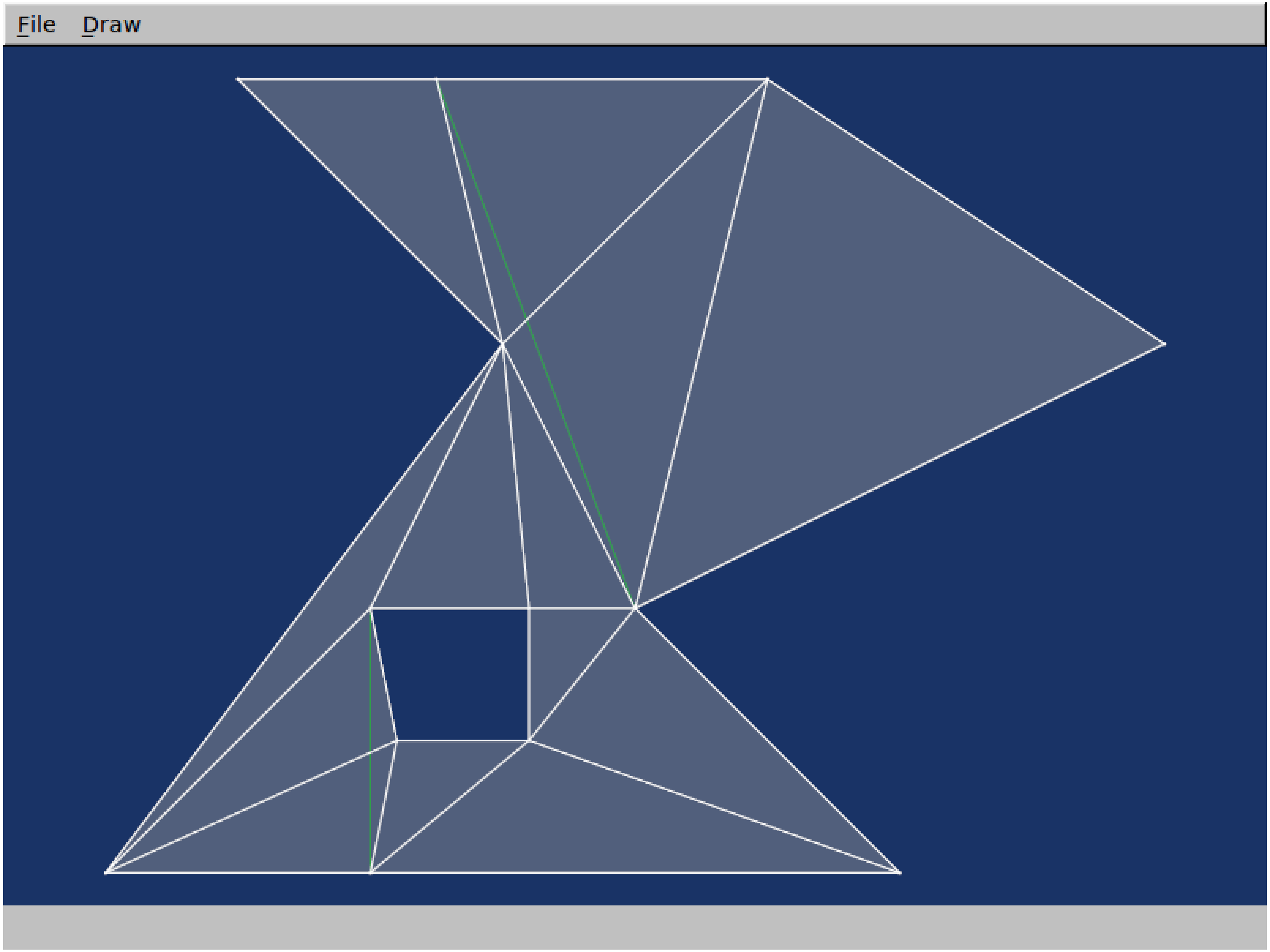
Delaunay Triangulations

- Example : Simple contour – triangulation



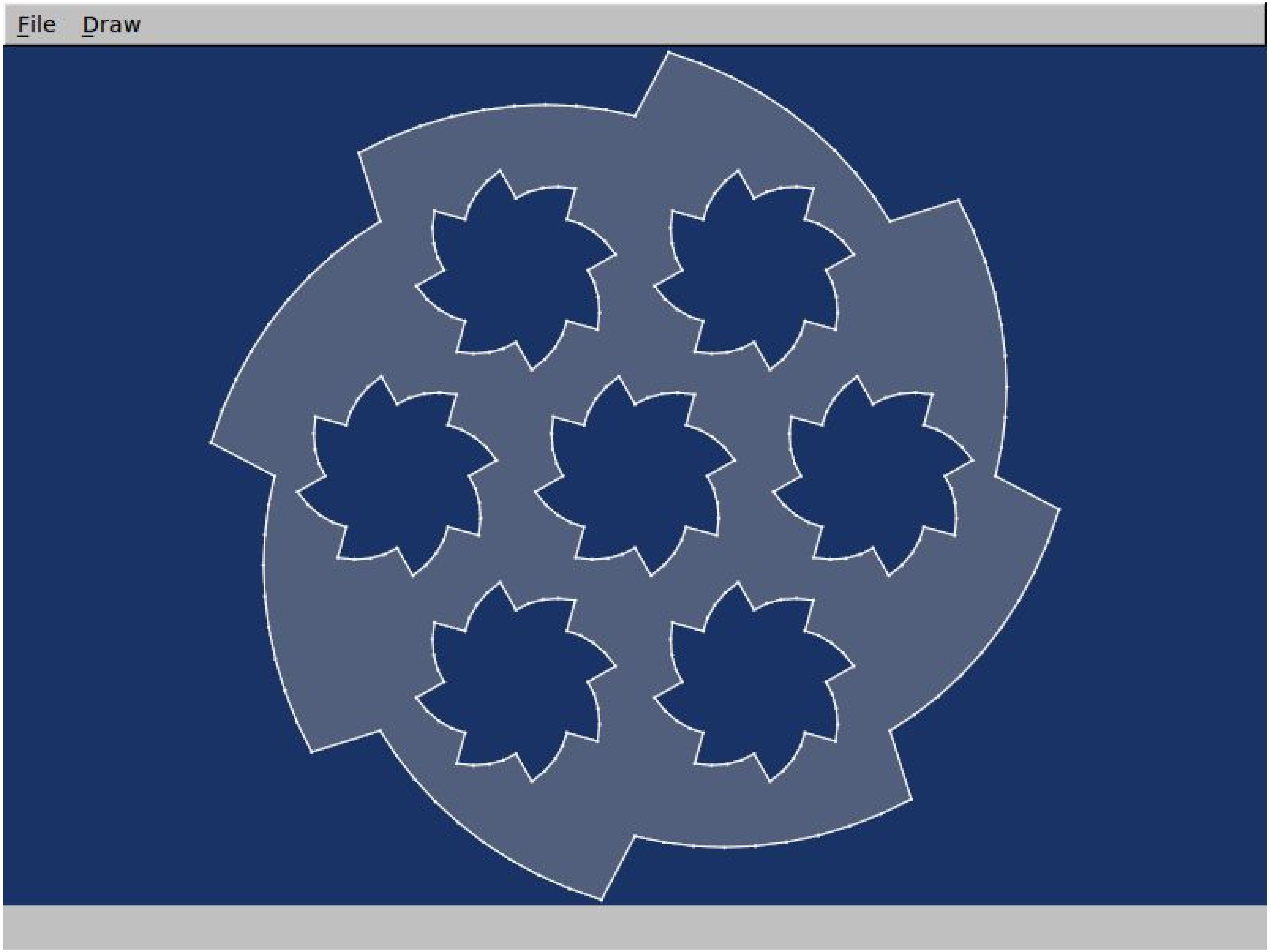
Delaunay Triangulations

- Example : Simple contour – Delaunay triangulation



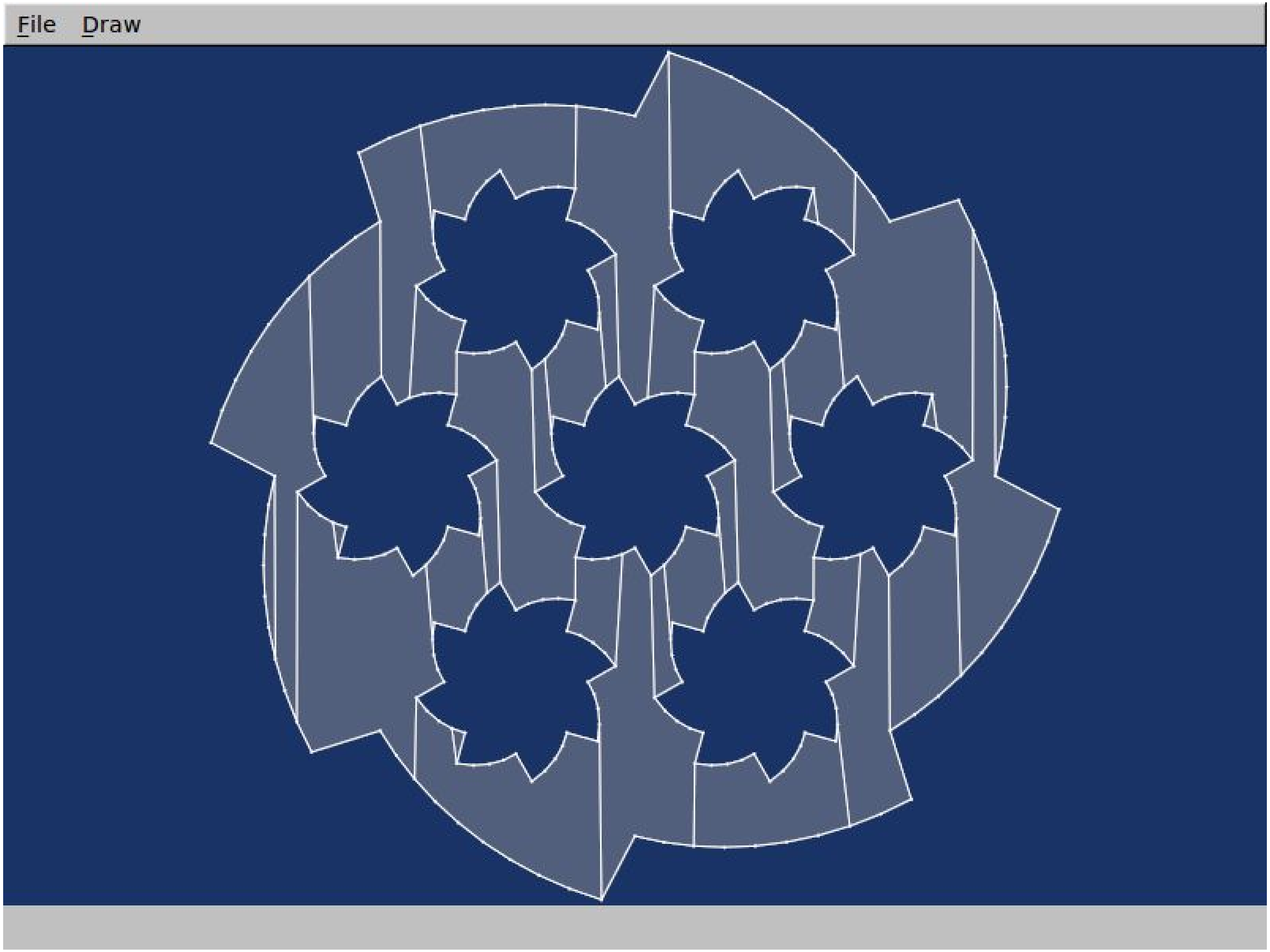
Delaunay Triangulations

- Example : More complex – Contour



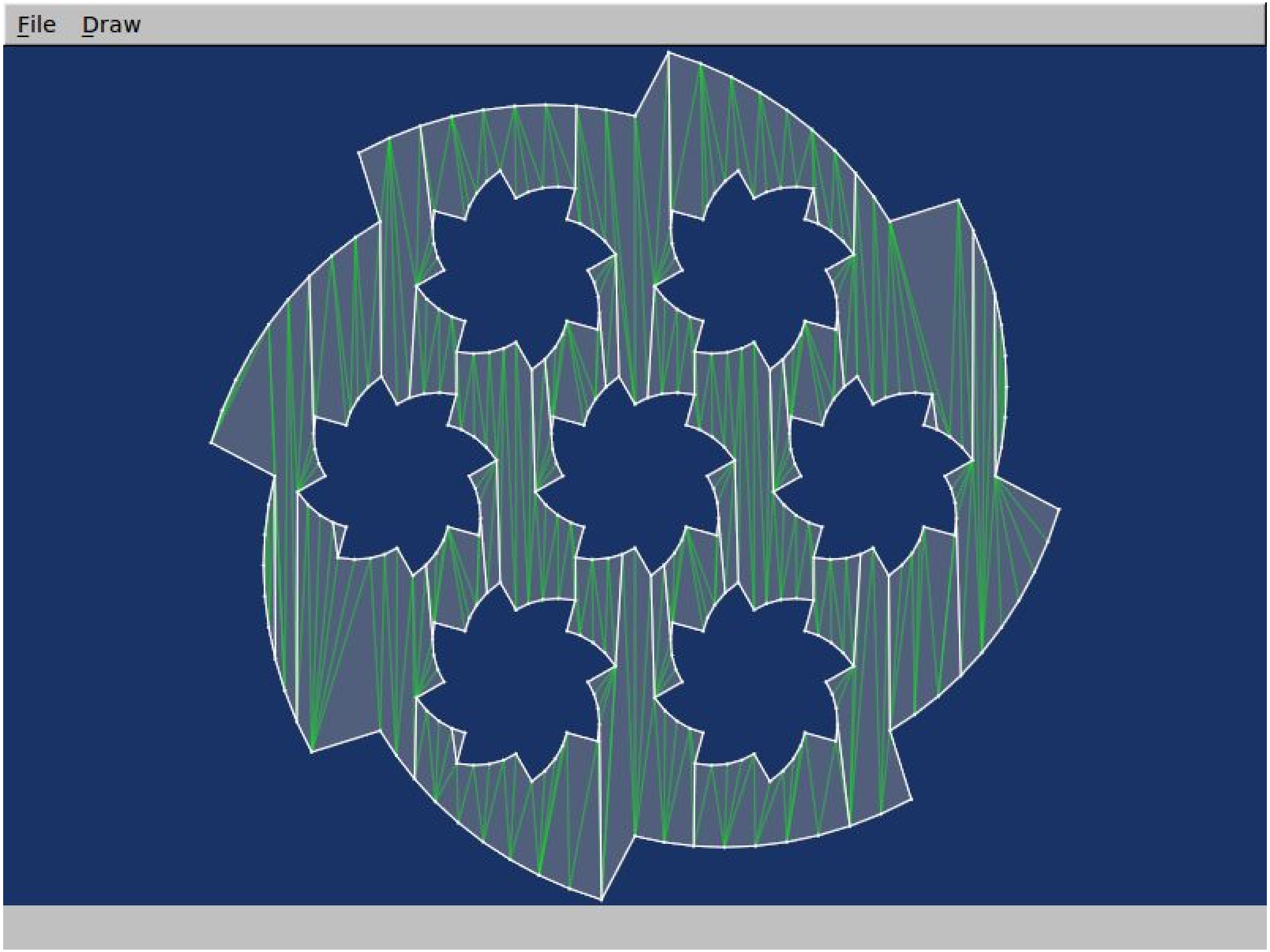
Delaunay Triangulations

- Example : More complex – Monotone polygons



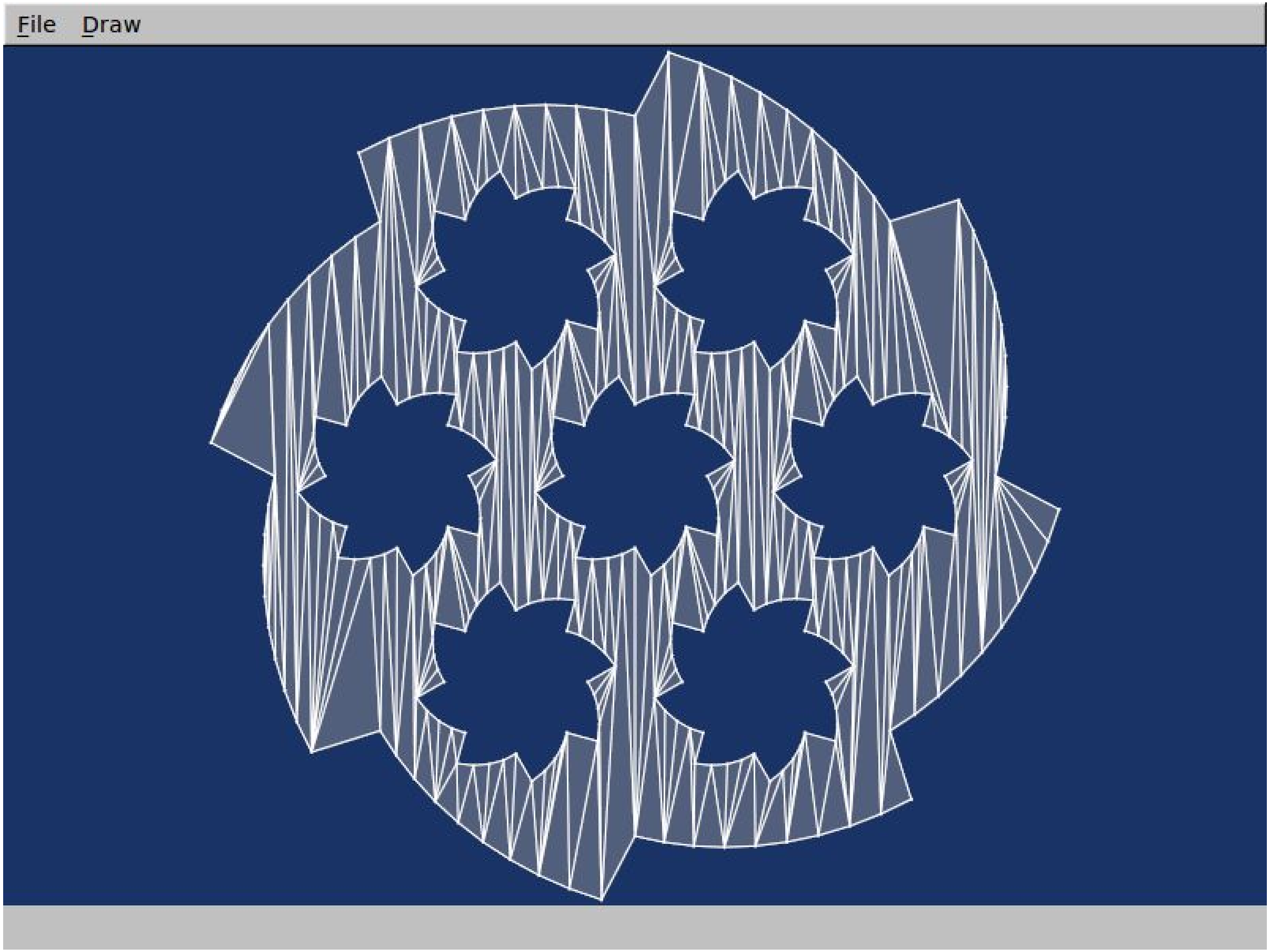
Delaunay Triangulations

- Example : More complex – Triangulation



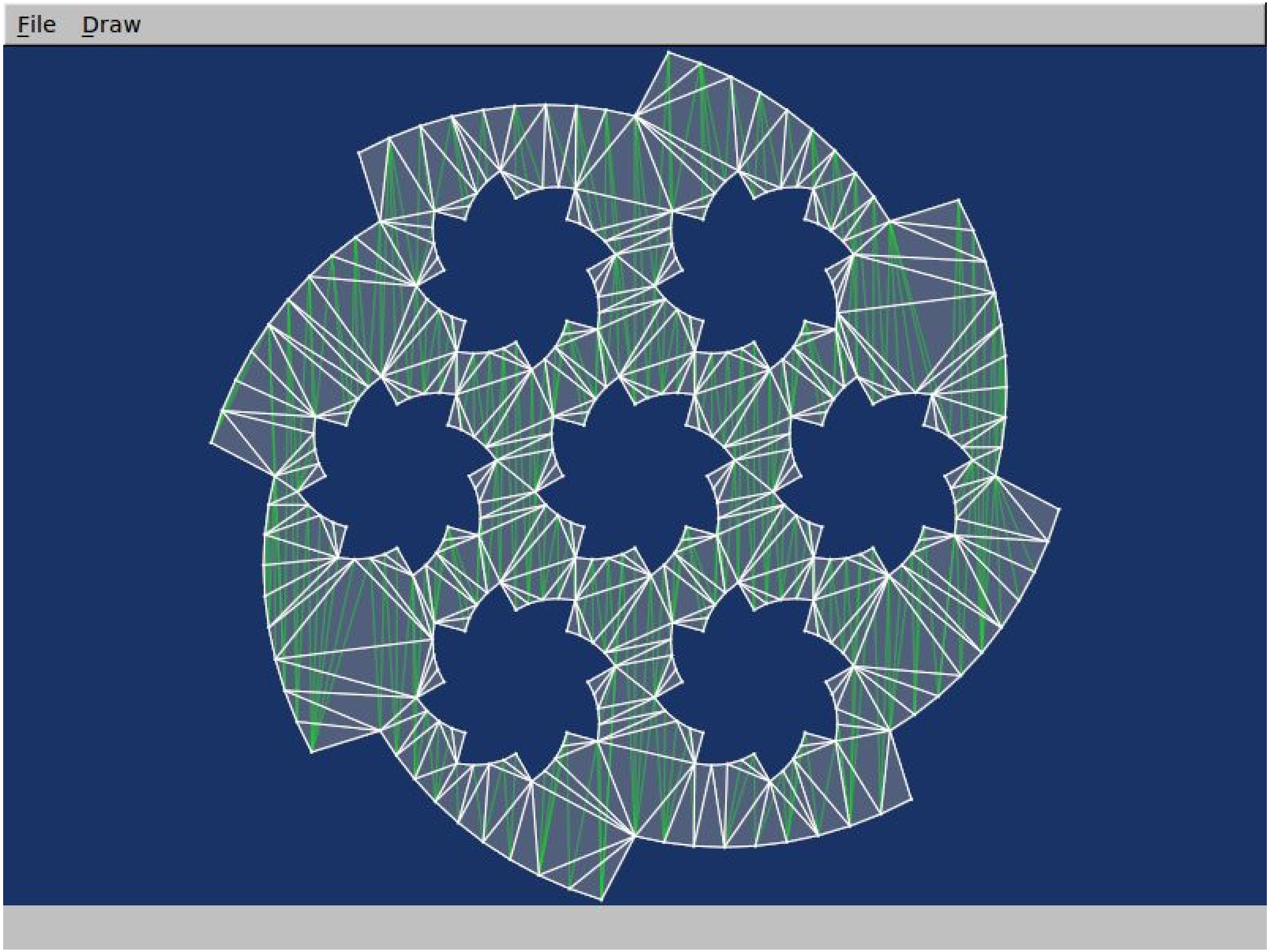
Delaunay Triangulations

- Example : More complex – Triangulation



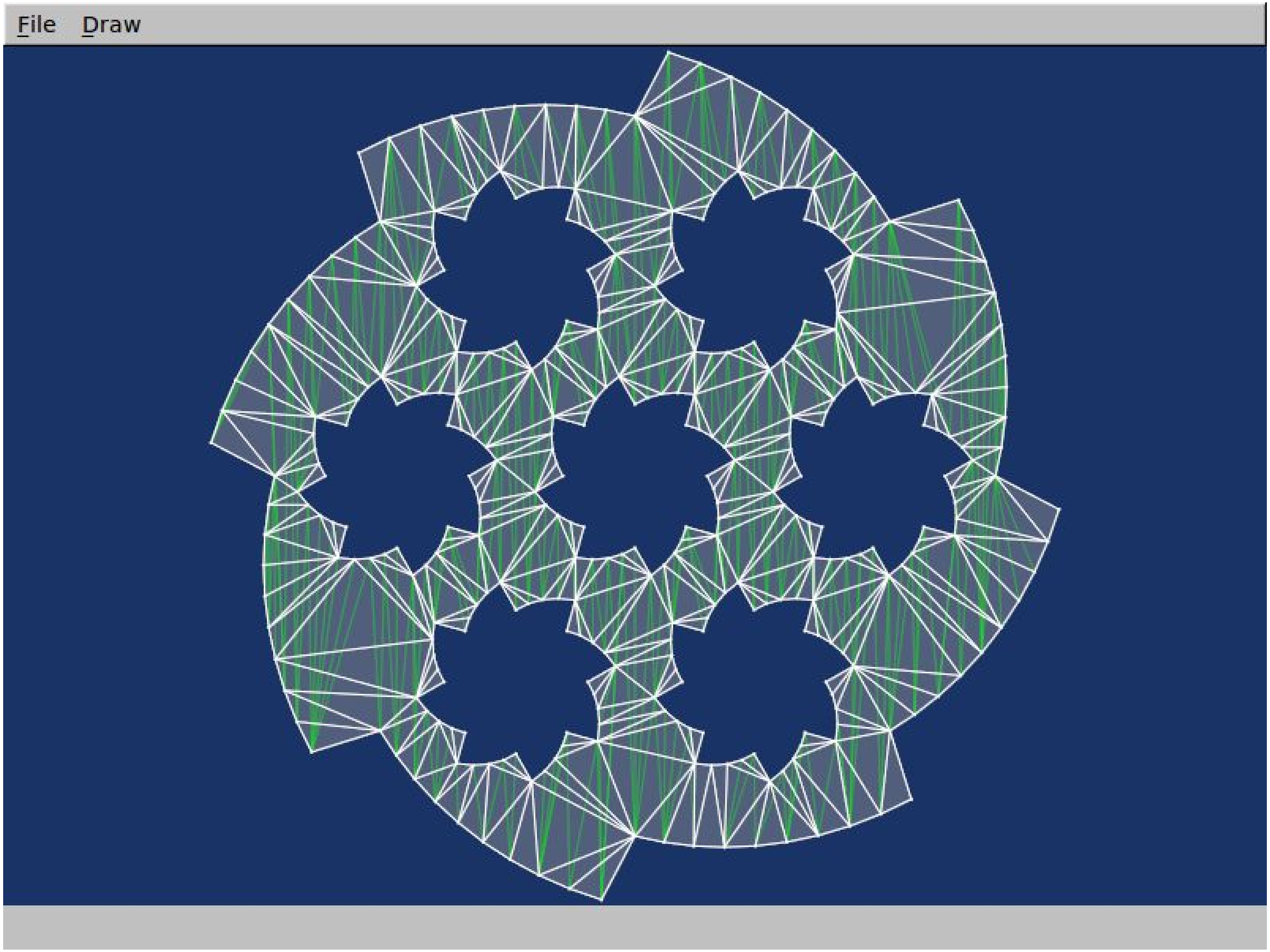
Delaunay Triangulations

- Example : More complex – Delaunay Tri. – path 1



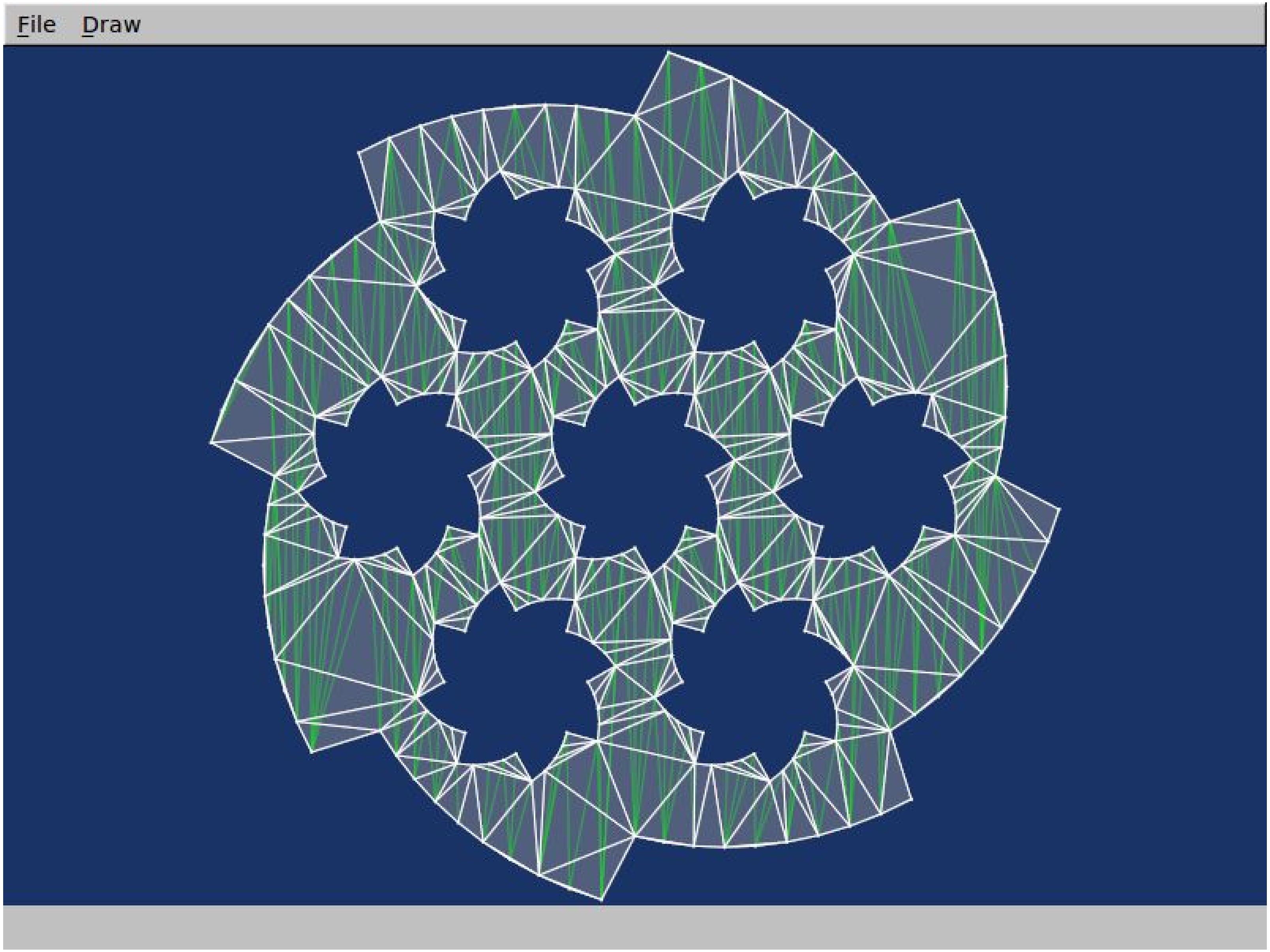
Delaunay Triangulations

- Example : More complex – Delaunay Tri. – path 2



Delaunay Triangulations

- Example : Quasi-minimum length – path 1



Delaunay Triangulations

- Example : Quasi-minimum length – path 2