

CAD&CG course

Masters in engineering

- **The course is given in 2023-2024 if there are enough students (5)**
- In that case, it will **not** be given in 2024-2025.  
Next session is 2025-2026

Please contact me if you are interested

E. Bechet

## Eric Béchet (it's me !)

- Engineering studies in Nancy (Fr.)
- Ph.D in Montréal (Can.)
- Academic career in Nantes as Post-doc then ass. Prof. in Metz (Fr.)  
then Liège...

## Alex Bolyn

- Assistant ( Ph.D. student ) at our university

Website <http://www.cgeo.ulg.ac.be/CADCG>

Contacts : [eric.bechet@uliege.be](mailto:eric.bechet@uliege.be)  
[a.bolyn@uliege.be](mailto:a.bolyn@uliege.be)

## Procedures

- Ex-Cathedra lectures (~2h )
- Practical sessions
  - Programming (~8 2h-sessions)
- Practical assessment : practical works ( $pw$ ) are rated.
- Project ( $pr$ )
- The practical sessions, and the project are all mandatory. The final mark  $m=1/3pw + 2/3pr$  , over 10 to pass & get the credits for the course.

## Procedures

- Lectures are given on wednesday, 1:45 PM –  
Lecture room is B52 +2/441 (or B52 +2/438 if few students)
- Begins today
- Practical work begins Sept. 27<sup>th</sup> after the course  
(check website for more dates & info)
- Any question : I am available mostly Fridays,  
appointment via email or phone : 04-366-9165  
Office : Bldg B52 +2/438

## Course Outline

- Introduction
  - Brief History of CAD Systems
- Generalities about Parametric Forms
  - Curves
  - Surfaces
- Some Practical Ways to represent Curves and Surfaces in CAD Systems
  - Cubic Splines
  - Bézier Curves
  - B-Splines

## Course Outline

- Solid Modeling in CAD systems
  - B-REP Models
- Computational Geometry
  - Convex Hull Problems
  - Geometric Search
  - Delaunay Triangulation
  - Mesh Generation

## Introduction

# CAD/CAM – Computer Aided Design / Computer Aided Manufacturing

- Development since the 1950s-1960s
  - Iso Schoenberg (at U-Wisconsin) – splines (40's)
  - First « interactive » CAD graphic engine at the MIT (I. Sutherland, Sketchpad) – 60's
  - James Ferguson (at Boeing) – splines (60's)
  - Paul De Casteljau (at Citroën) – Bézier curves (50's)
  - Pierre Bézier (at Renault) – " "
  - Carl De Boor (at GM / U-Wisconsin) – B-Splines (70's)
    - Catia, Unigraphics (now Siemens NX) date back to the mid-70's, emergence of 3d modeling kernels on mainframes.
- Huge expansion in the 80's with cheap personal computers becoming ubiquitous.



## Introduction

- Sketchpad (Ivan Sutherland) – 1963



## Introduction

- Computational geometry
  - As old as ways to represent geometrical entities on a computer
  - Focus in this course
    - Use CAD as an initial geometrical database
    - Define operations in this setting and show how to perform them efficiently – or not !
    - Focus is made on applications in scientific computing, e.g. mesh generation, ...

## Introduction

- Old mathematical and theoretical bases
- Practical framework has been developed because of industrial needs.
- In return, it has contributed at the theoretical level.

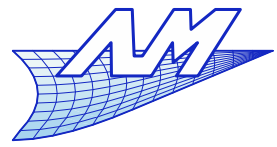
## Parametric representation

## Parametric representation

### Use of scalar parameters to “walk” on the curve/surface

- The number of those scalars determine the dimension of the entity and the mapping from one space (the parametric space to the other (usually cartesian) space.
  - 0 – Point , 1 ( $u$ ) – Curve , 2 ( $u, v$ ) – Surface  
3 ( $u, v, w$ ) – Volume ( transformation used for finite elements)
- We associate to those scalars some function for each of the space coordinates (  $x, y$  in the plane;  $x, y, z$  in 3D)

$$\vec{P}(u) = \begin{cases} x = f(u) \\ y = g(u) \\ z = h(u) \end{cases} \quad \text{or} \quad \vec{P}(u, v) = \begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases}$$



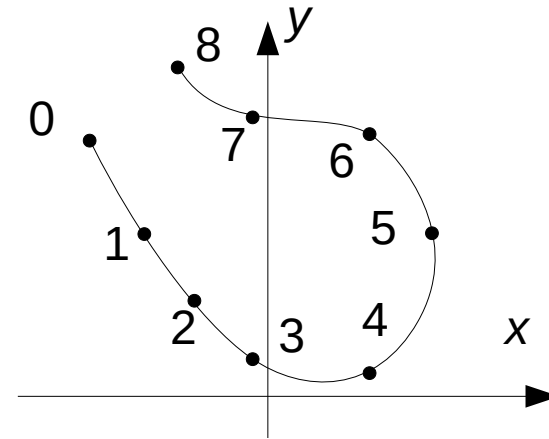
## Parametric representation

### Parametric curves

## Parametric representation

For a curve, the representation takes the following form:

$$\vec{P}(u) = \begin{cases} x = f(u) \\ y = g(u) \\ z = h(u) \end{cases}$$



$u$  is a real parameter.

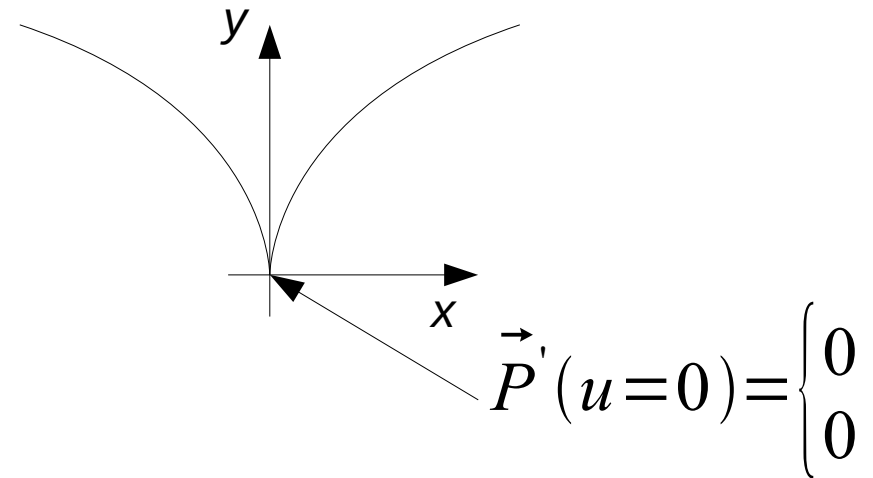
- We obtain all the points of the curve by varying  $u$ .
- The parametrization is not unique !
- The parameter  $u$  can be limited (limits of the curve)

## Parametric representation

### Continuity of the parametrization

- A parametric curve is said to be  $C_k$  if the parametrization  $P(u)$  is  $C_k$  (i.e. the  $k^{\text{th}}$  derivative is continuous)
- However : a parametric curve of class  $C_\infty$  may have an angular point... (i.e. some continuity of the parametrization does not necessarily involve the same **geometric** continuity of the resulting curve !)

$$\vec{P}(u) = \begin{cases} x = u^3 \\ y = u^2 \end{cases}$$





## Parametric representation

### Regularity of the parametrization

- A parametric curve  $P(u)$  is *regular* if the first derivative  $P'(u)$  does not vanish at any place on all the interval of definition.
- On the contrary, points where  $P'(u)$  vanishes, are called *singular* points.
- A given curve can admit two parametric forms such that one is regular and the other one isn't...
  - Example : curve  $y=x^2$  with two parametric forms  $P(u)$  and

$$\vec{P}(u) = \begin{cases} x = u \\ y = u^2 \end{cases} \quad \vec{P}'(u=0) = \begin{cases} 1 \\ 0 \end{cases} \quad \vec{Q}(v) = \begin{cases} x = v^3 \\ y = v^6 \end{cases} \quad \vec{Q}'(v=0) = \begin{cases} 0 \\ 0 \end{cases}$$

## Parametric representation

### Regularity and continuity of the parametrization

- A curve of geometric continuity  $G_k$  can be described by a parametrization that is non-  $C_k$
- A curve of parametrization  $C_k$  may not be  $G_k$
- A curve admitting singular points can be  $G_k$  and/or  $C_k$ .
- A regular curve is not necessary  $C_k$  and/or  $G_k$ .

To conclude :

- Parametric continuity, geometric continuity, and regularity are all relatively independent notions!

## Parametric representation

### $C_k$ -equivalent parametric forms

- Two parametric forms  $P(u)$  and  $Q(v)$  of the same curve are  $C_k$  - equivalents if there is a bijective function  $\varphi$  of class  $C_k$  whose reciprocal is also of class  $C_k$  and such that:

$$v = \varphi(u) \quad \vec{P}(u) = \vec{Q}(\varphi(u))$$

## Parametric representation

### Length of a regular curve

- It is a measure that is independent of the parametrization (provided that those are  $C_1$ -equivalent)

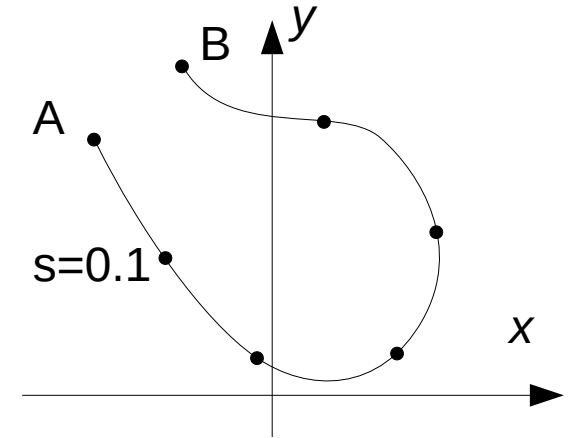
$$L = \int_a^b \left| \vec{P}'(u) \right| du = \int_{c=\varphi(a)}^{d=\varphi(b)} \left| \vec{Q}'(v) \right| dv$$

## Parametric representation

### Natural parametrization

- It is a parametrization  $s$  such that :

$$L(AB) = s(B) - s(A)$$



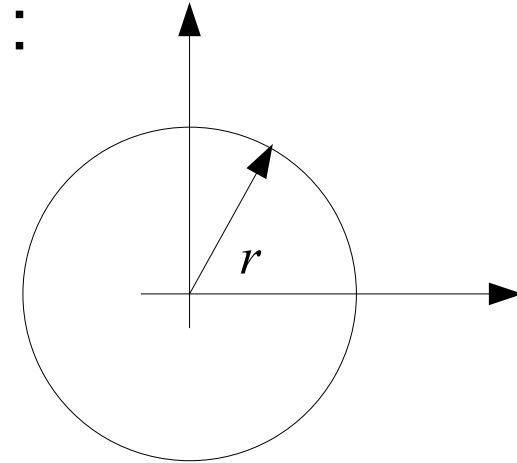
- $s$  is also named *curvilinear abscissa along the curve*.
- It can be built from any regular parametrization  $u$  of the curve :

$$s(u) = \int_A^u \left| \vec{P}'(t) \right| dt$$

## Parametric representation

- Parametrization of a circle in the plane :

$$\begin{cases} x = r \cos u \\ y = r \sin u \end{cases}, \quad u \in [0, 2\pi[$$



- Natural parametrization of the same circle...

$$\begin{cases} x = r \cos \frac{s}{r} \\ y = r \sin \frac{s}{r} \end{cases}, \quad s \in [0, 2\pi r[ \quad s = r u$$

## Parametric representation

### Differential geometry for parametric curves

- Position  $P$  : 
$$P(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} \quad P' = \frac{dP}{du} \quad P'' = \frac{d^2 P}{du^2}$$

- Tangent vector  $T$  :

$$T(u) = \frac{dP}{ds} = \frac{dP}{du} \cdot \frac{du}{ds} = \frac{P'}{|P'|}$$

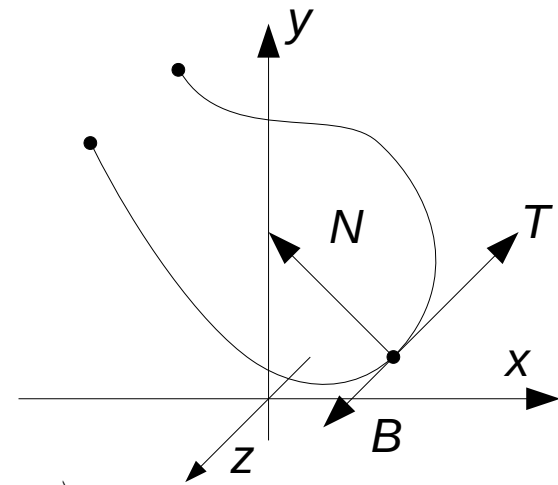
- Normal vector  $N$  :

$$N(u) = \frac{Norm(u)}{|Norm(u)|} \quad \text{with} \quad Norm(u) = P'' - (P'' \cdot T)T$$

(It is not defined when  $Norm(u) = 0$  !)

- Binormal vector  $B$  :

$$B(u) = T(u) \times N(u)$$

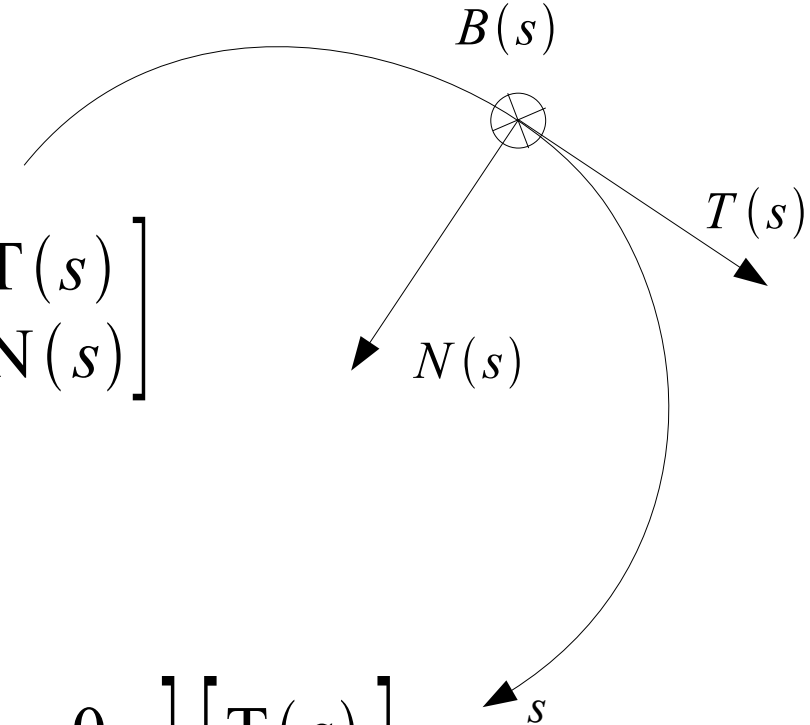


## Parametric representation

### Frenet's frame and equations

- 2D 
$$\begin{bmatrix} \frac{d T(s)}{ds} \\ \frac{d N(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}$$

- 3D 
$$\begin{bmatrix} \frac{d T(s)}{ds} \\ \frac{d N(s)}{ds} \\ \frac{d B(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$



Works only with  $s$  : the curvilinear abscissa !



## Parametric representation

- Curvature  $\kappa$  :

$$\kappa(u) = N(s) \cdot \frac{dT}{ds} = \frac{T'(u) \cdot N(u)}{|P'(u)|}$$

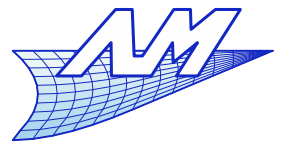
$$|\kappa| = \left| \frac{dT}{ds} \right| = \frac{|P' \times P''|}{|P'|^3}$$

Curvature vector  
 $= N(u) \kappa(u)$

- Torsion  $\tau$ :

$$\tau(u) = \frac{N'(u) \cdot B(u)}{|P'(u)|}$$

$$\tau = \frac{\left( \frac{d^2 T}{ds^2}, \frac{dT}{ds}, T \right)}{\kappa^2} = \frac{(P''', P'', P')}{|P' \times P''|^2}$$



## Parametric representation

Parametric surfaces

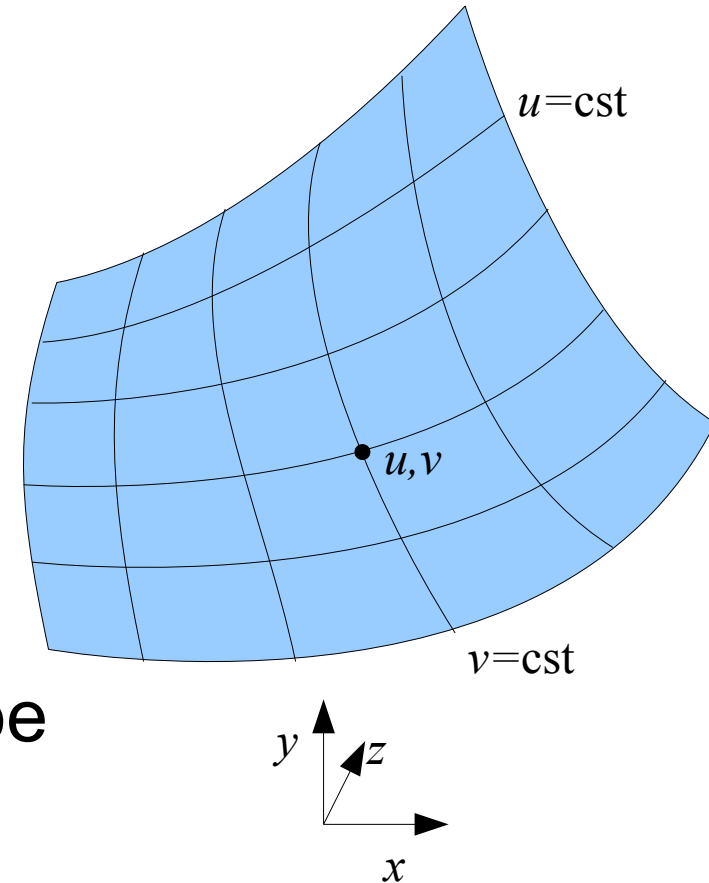
## Parametric representation

A parametric surface is represented as :

$$\vec{P}(u, v) = \begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases}$$

$u, v$  are two real parameters

- All the points of the surface can be obtained by varying  $u$  and  $v$ .
- Again, the parametrization is not unique !
- The interval of definition is limited (limits the surface)



## Parametric representation

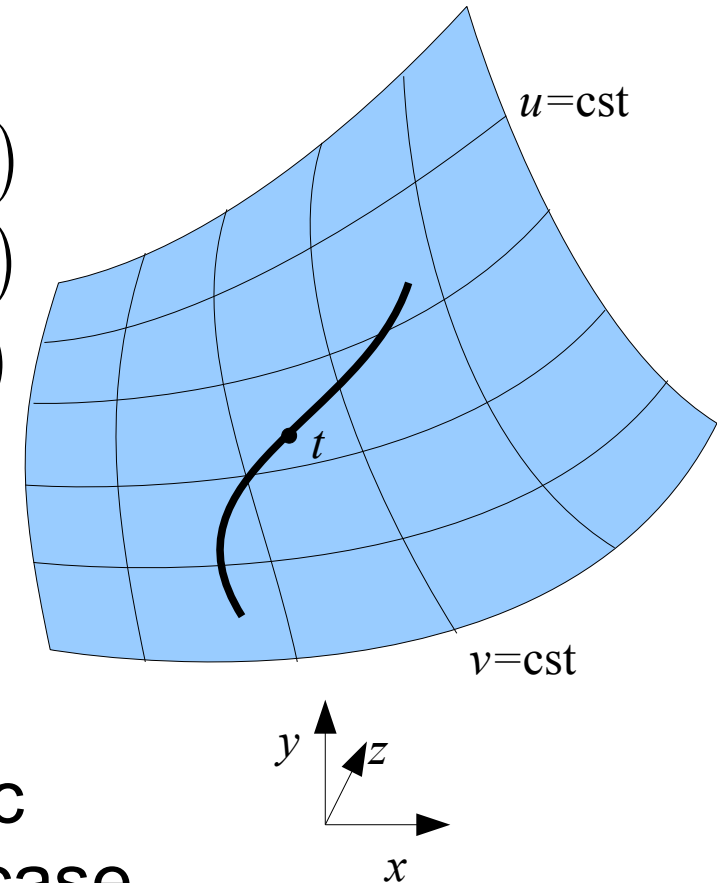
A curve can be defined on the surface

Parametric space

« Ambient » space

$$\vec{\Gamma}^{uv}(t) : \begin{cases} u = u(t) \\ v = v(t) \end{cases} \longrightarrow \vec{P}(u, v) : \begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases}$$

$$\vec{\Gamma}(t) : \begin{cases} x = f(u(t), v(t)) \\ y = g(u(t), v(t)) \\ z = h(u(t), v(t)) \end{cases}$$



- The equations seen for parametric curves are valid in this particular case.

## Parametric representation

### Regularity and continuity of the parametrization

- A parametric surface is of class  $C_k$  if the mapping  $P(u, v)$  onto  $\mathbb{R}^3$  is of class  $C_k$ .
- A parametrization is *regular* if and only if (iff)

$$\frac{\partial \vec{P}}{\partial u}(u_0, v_0) \times \frac{\partial \vec{P}}{\partial v}(u_0, v_0) \neq \vec{0} \quad \forall (u_0, v_0) \in D \subset \mathbb{R}^2$$

- The points on the parametric space for which it is not the case are called *singular* points.
- $C_k$ -equivalent parametric forms follow the same definitions as for curves

## Parametric representation

### Differential geometry for parametric surfaces

- Position  $P$  :  $\vec{P}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$      $\vec{P}^u = \frac{\partial \vec{P}}{\partial u}$      $\vec{P}^{uv} = \frac{\partial^2 \vec{P}}{\partial u \partial v}$  ...

- Unit tangent vectors  $T^u$  and  $T^v$  :

$$\vec{T}^u(u, v) = \frac{\partial P}{\partial u} \cdot \left| \frac{\partial P}{\partial u} \right|^{-1} = \frac{P^u}{|P^u|} \quad \vec{T}^v(u, v) = \frac{\partial P}{\partial v} \cdot \left| \frac{\partial P}{\partial v} \right|^{-1} = \frac{P^v}{|P^v|}$$

- These vectors are, in general, not orthogonal !
- Tangent plane (parametric form)

$$\vec{P}t_{(u_0, v_0)}(a, b) = \vec{P}(u_0, v_0) + a \cdot \vec{T}^u(u_0, v_0) + b \cdot \vec{T}^v(u_0, v_0)$$

or

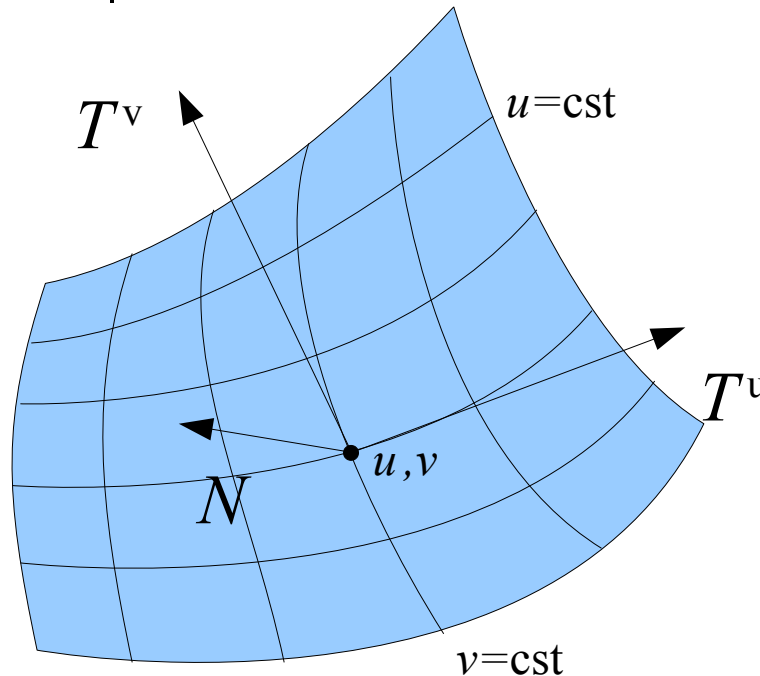
$$(a, b) \in \mathbb{R}^2$$

$$\vec{P}t_{(u_0, v_0)}(a, b) = \vec{P}(u_0, v_0) + a \cdot \vec{P}^u(u_0, v_0) + b \cdot \vec{P}^v(u_0, v_0)$$

## Parametric representation

- Normal vector  $N$  :

$$N(u, v) = \frac{Norm(u, v)}{|Norm(u, v)|} \text{ with } Norm(u, v) = T^u \times T^v \text{ or } P^u \times P^v$$

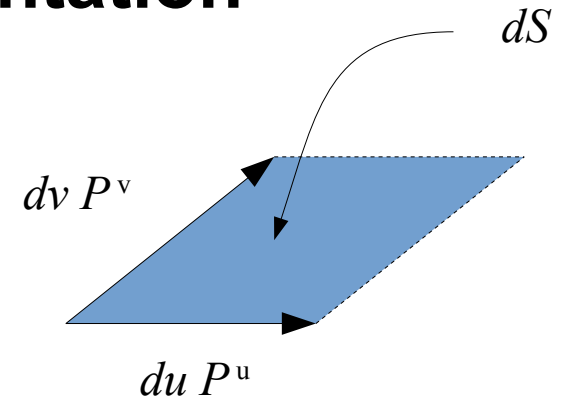


## Parametric representation

- Area of a surface

$$A = \iint_{\Omega} dS$$

$$dS = |du \cdot P^u \times dv \cdot P^v| = |P^u \times P^v| dudv$$



### 1<sup>st</sup> fundamental form

- Other notation of the area of a surface

$$|\vec{a} \times \vec{b}|^2 = (\vec{a} \cdot \vec{a}) \cdot (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2 \quad \leftarrow \text{Lagrange's identity}$$

$$dS = \sqrt{(eg - f^2)} dudv \quad \text{with } e = P^u \cdot P^u, \quad f = P^u \cdot P^v, \quad g = P^v \cdot P^v$$

$$A = \iint_D \sqrt{(eg - f^2)} dudv$$



## Parametric representation

- Length of a curve on a surface

$$\vec{P}(u, v) : \begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases} \quad \begin{aligned} P^u &= \frac{\partial P(u, v)}{\partial u} \\ P^v &= \frac{\partial P(u, v)}{\partial v} \end{aligned}$$

$$\vec{\Gamma}^{uv}(t) : \begin{cases} u = u(t) \\ v = v(t) \end{cases} \quad \vec{\Gamma}(t) : \begin{cases} x = f(u(t), v(t)) \\ y = g(u(t), v(t)) \\ z = h(u(t), v(t)) \end{cases} \quad \Gamma' = \frac{dP(u(t), v(t))}{dt}$$

$$\Gamma' = \frac{\partial P(u(t), v(t))}{\partial u} \frac{du}{dt} + \frac{\partial P(u(t), v(t))}{\partial v} \frac{dv}{dt}$$

Derivative of the curve expressed  
in the parametric space of the surface

Derivative of the surface (in 3D)

$$\Gamma'(t) = \vec{u}'(t) P^u(u(t), v(t)) + v'(t) P^v(u(t), v(t))$$

## Parametric representation

$$L = \int_a^b \left| \vec{\Gamma}'(t) \right| dt = \int_a^b \sqrt{\left| \vec{\Gamma}'(t) \right|^2} dt$$

$$\Gamma'(t) = u'(t) P^u(u(t), v(t)) + v'(t) P^v(u(t), v(t))$$

$$|\Gamma'(t)|^2 = e u'(t)^2 + 2f u'(t)v'(t) + g v'(t)^2$$

$$\text{with } e = P^u \cdot P^u, \quad f = P^u \cdot P^v, \quad g = P^v \cdot P^v$$

## Parametric representation

- If we set

$$ds = \sqrt{e u'(t)^2 + 2 f u'(t) v'(t) + g v'(t)^2} dt$$

that is equivalent to :  $ds = \sqrt{e du^2 + 2 f dudv + g dv^2}$

( we have  $L = \int_{s(a)}^{s(b)} ds = s(b) - s(a)$  )

, we get a quadratic form :

$$e u'(t)^2 + 2 f u'(t) v'(t) + g v'(t)^2 = \begin{pmatrix} u'(t) & v'(t) \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}$$

$$L = \int_a^b \sqrt{\begin{pmatrix} u'(t) & v'(t) \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}} dt$$

## Parametric representation

- Angle between two curves ...

$$\Gamma_1'(t_1) \cdot \Gamma_2'(t_2) = |\Gamma_1'(t_1)| |\Gamma_2'(t_2)| \cos \alpha = \begin{pmatrix} u_1'(t_1) & v_1'(t_1) \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u_2'(t_2) \\ v_2'(t_2) \end{pmatrix}$$

$$\cos \alpha = \frac{\begin{pmatrix} u_1'(t_1) & v_1'(t_1) \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u_2'(t_2) \\ v_2'(t_2) \end{pmatrix}}{\sqrt{\begin{pmatrix} u_1'(t_1) & v_1'(t_1) \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u_1'(t_1) \\ v_1'(t_1) \end{pmatrix} \begin{pmatrix} u_2'(t_2) & v_2'(t_2) \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u_2'(t_2) \\ v_2'(t_2) \end{pmatrix}}}$$

## Parametric representation

- The first fundamental form is the application :

$$\varphi_1(d\Gamma_1^{uv}, d\Gamma_2^{uv}) = \begin{pmatrix} du_1 & dv_1 \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} du_2 \\ dv_2 \end{pmatrix} = \begin{pmatrix} du_1 & dv_1 \end{pmatrix} M_1 \begin{pmatrix} du_2 \\ dv_2 \end{pmatrix}$$

with  $e = P^u \cdot P^u$  ,  $f = P^u \cdot P^v$  ,  $g = P^v \cdot P^v$

- It is a symmetric bilinear form that allows to measure real distances from variations in the parametric space...

- The matrix  $M_1$  is a representation of the metric tensor, often noted  $g_{ij}, i, j = u, v$

- $M_1$  also is related to the Jacobian matrix  $J = \begin{pmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \\ \partial z / \partial u & \partial z / \partial v \end{pmatrix}$  of the transformation  $(u, v) \rightarrow (x, y, z)$  (it is  $J^T J$ ).

$$L = \int_a^b \sqrt{\varphi(d\Gamma^{uv}, d\Gamma^{uv})} dt \quad \cos \alpha = \frac{\varphi(d\Gamma_1^{uv}, d\Gamma_2^{uv})}{\sqrt{\varphi(d\Gamma_1^{uv}, d\Gamma_1^{uv}) \varphi(d\Gamma_2^{uv}, d\Gamma_2^{uv})}}$$

$$A = \iint_D \sqrt{\det M_1} dudv \quad \left( = \iint_D \det J dudv \text{ under some conditions} \right)$$

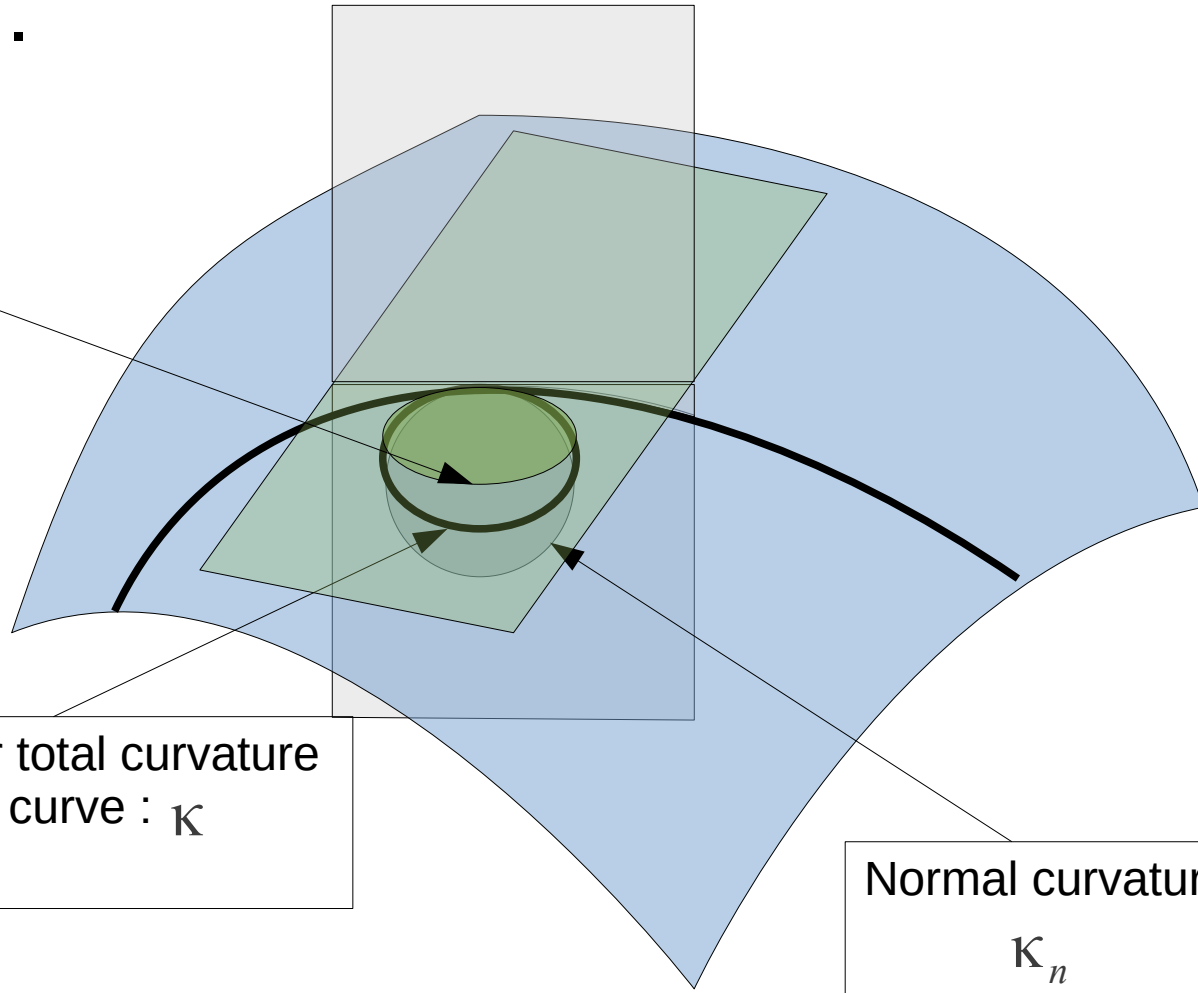
# Parametric representation

- Curvatures...

Geodesic curvature  
or relative curvature  
w/r to the surface :  
 $\kappa_g$

Simple or total curvature  
of the 3D curve :  $\kappa$

Normal curvature :  
 $\kappa_n$



## Parametric representation

- Curvature vectors

$$\mathbf{k} = \frac{d\mathbf{T}}{ds} = \kappa \cdot \mathbf{n} = \mathbf{k}_n + \mathbf{k}_g \quad (\text{e.g. from Frenet's relations})$$

$$\mathbf{k}_n = \kappa_n \cdot \mathbf{N} \quad \mathbf{k}_g = \kappa_g \cdot \mathbf{G}$$

- Necessarily, we do have :  $\mathbf{N} \cdot \mathbf{T} = 0$

- Differentiating yields :

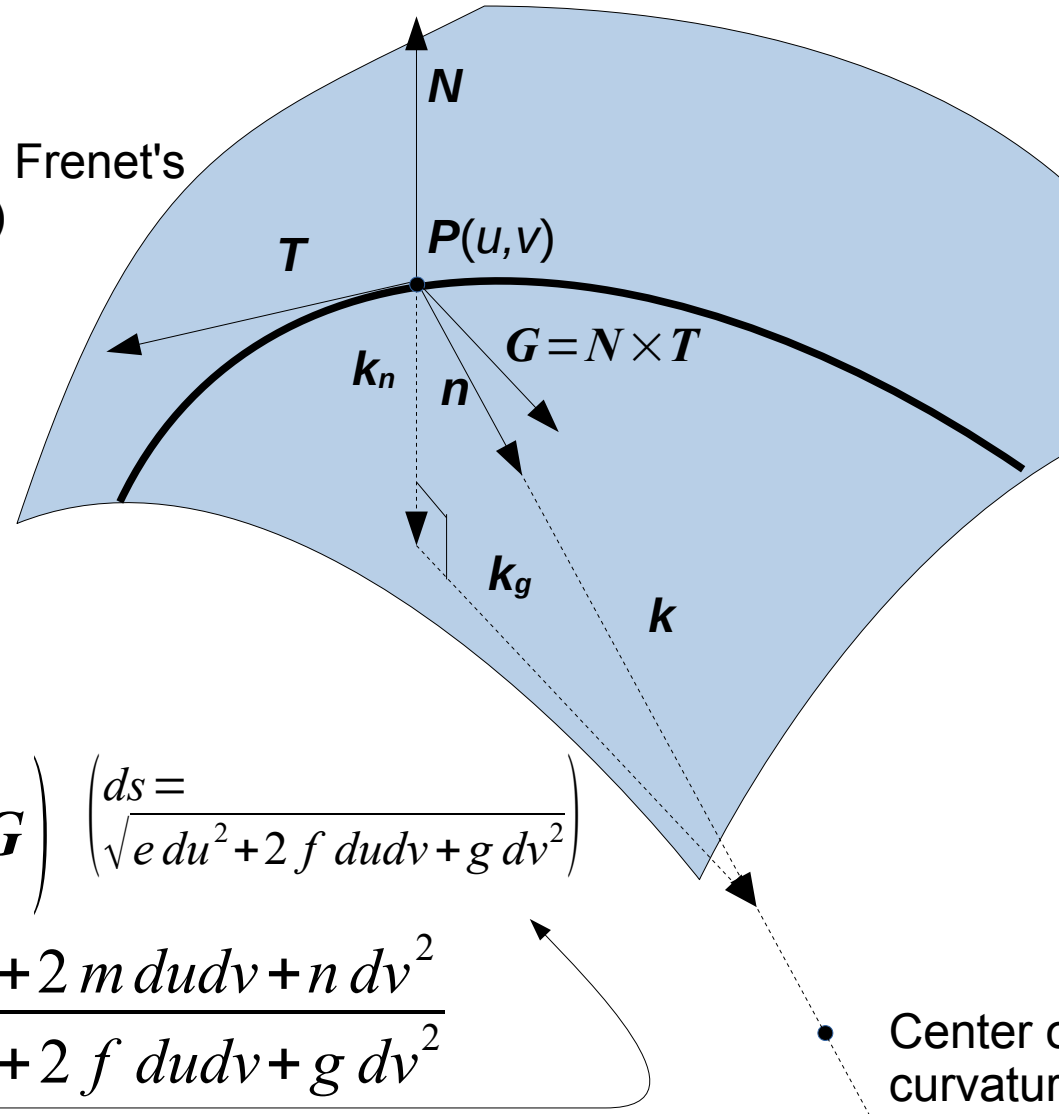
$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{N} + \mathbf{T} \cdot \frac{d\mathbf{N}}{ds} = 0$$

$$\kappa_n = \frac{d\mathbf{T}}{ds} \cdot \mathbf{N} = -\mathbf{T} \cdot \frac{d\mathbf{N}}{ds} \quad \left( \kappa_g = \frac{d\mathbf{T}}{ds} \cdot \mathbf{G} \right) \quad \left( ds = \frac{1}{\sqrt{e du^2 + 2f dudv + g dv^2}} \right)$$

$$= -\frac{d\mathbf{P}(u, v)}{ds} \cdot \frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{P} \cdot d\mathbf{N}}{ds^2} = \frac{l du^2 + 2m dudv + n dv^2}{e du^2 + 2f dudv + g dv^2}$$

with

$$l = -\frac{dP}{du} \cdot \frac{dN}{du} \quad m = -\frac{1}{2} \left( \frac{dP}{du} \cdot \frac{dN}{dv} + \frac{dP}{dv} \cdot \frac{dN}{du} \right) \quad n = -\frac{dP}{dv} \cdot \frac{dN}{dv}$$



Center of curvature

## Parametric representation

$$l = -\frac{dP}{du} \cdot \frac{dN}{du} \quad m = -\frac{1}{2} \left( \frac{dP}{du} \cdot \frac{dN}{dv} + \frac{dP}{dv} \cdot \frac{dN}{du} \right) \quad n = -\frac{dP}{dv} \cdot \frac{dN}{dv}$$

- This can be rewritten by noting that

$$\frac{dP}{du} \cdot N = 0 \quad \text{and} \quad \frac{dP}{dv} \cdot N = 0$$

, and integrating by parts  $UV = \int U dV + \int V dU$

where e.g. for  $l$  :  $U = \frac{dP}{du} \quad dV = \frac{dN}{du} \cdot du$

$$l = \frac{d^2 P}{du^2} \cdot N \quad m = \frac{d^2 P}{du dv} \cdot N \quad n = \frac{d^2 P}{dv^2} \cdot N$$



## Parametric representation

### Curvatures and the second fundamental form

- The second fundamental form is the application

$$\varphi_2(d\Gamma_1^{uv}, d\Gamma_2^{uv}) = \begin{pmatrix} du_1 & dv_1 \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} du_2 \\ dv_2 \end{pmatrix} = \begin{pmatrix} du_1 & dv_1 \end{pmatrix} M_2 \begin{pmatrix} du_2 \\ dv_2 \end{pmatrix}$$

with  $l = N \cdot P^{uu}$ ,  $n = N \cdot P^{vv}$ ,  $m = N \cdot P^{uv}$

- Normal curvature of a curve on a surface :

$$\kappa_n = \frac{\varphi_2(d\Gamma^{uv}, d\Gamma^{uv})}{ds^2} \qquad \kappa_n = \frac{\varphi_2(d\Gamma^{uv}, d\Gamma^{uv})}{\varphi_1(d\Gamma^{uv}, d\Gamma^{uv})}$$

- Geodesic curvature (curvature in the tangent plane)
  - Measures the deviation of the curve in comparison with a geodesic

## Parametric representation

### Definition of the geodesic

- It is a curve with a minimal length on a surface.
  - In the plane, those are lines
  - On a sphere, great circles
  - By definition, the geodesic has no geodesic curvature.

Relations between « simple » curvature, geodesic curvature and normal curvature is obtained from the curvature vectors

$$\mathbf{k} = \mathbf{k}_n + \mathbf{k}_g \quad \mathbf{k}_n = \kappa_n \cdot \mathbf{N} \quad \mathbf{k} = \kappa \cdot \mathbf{n} \quad \mathbf{k}_g = \kappa_g \cdot (\mathbf{N} \times \mathbf{T}) = \kappa_g \cdot \mathbf{G}$$

$$\kappa^2 = \kappa_n^2 + \kappa_g^2$$

The geodesic curvature is therefore readily obtained from the others (except for the sign, which is same as  $\frac{d\mathbf{T}}{dt} \cdot \mathbf{G}$ )

$$\kappa_g^2 = \kappa^2 - \kappa_n^2$$

## Parametric representation

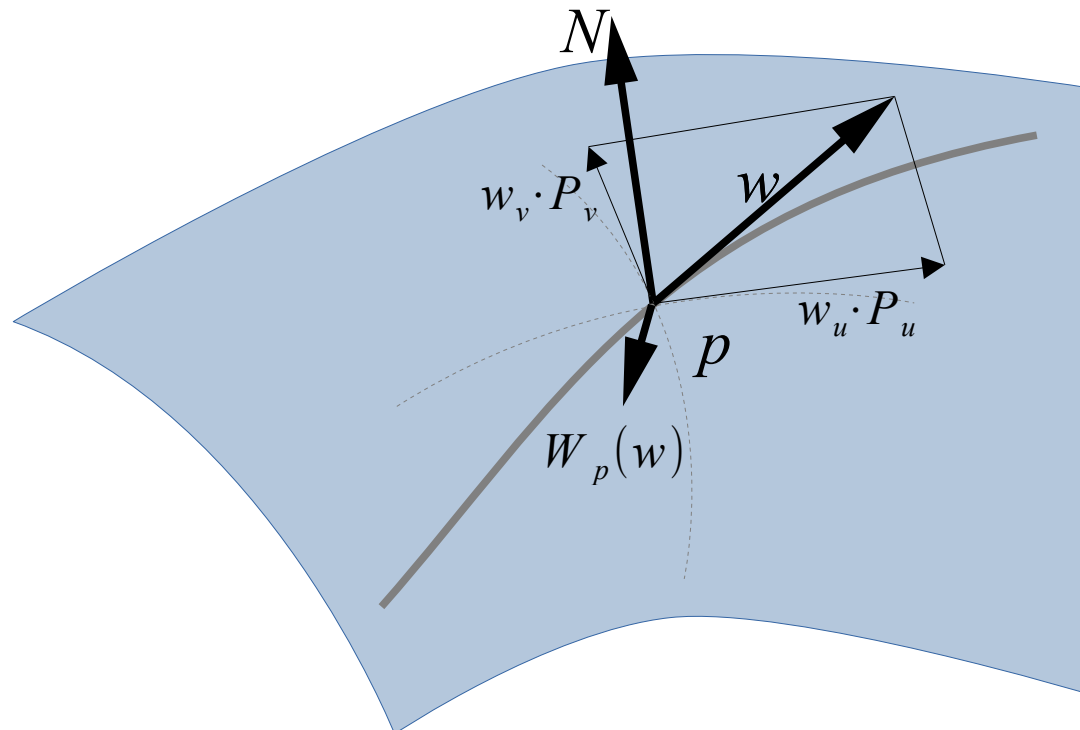
- A circle of radius  $R$  in the plane
  - Curvature  $\kappa = 1/R$
  - Geodesic curvature  $\kappa_g = 1/R$
  - Normal curvature  $\kappa_n = 0$
- A equatorial circle on a sphere of radius  $R$ 
  - Curvature  $\kappa = 1/R$
  - Geodesic curvature  $\kappa_g = 0$
  - Normal curvature  $\kappa_n = 1/R$
- In the previous slides,  $\kappa$  is the curvature of the curve in 3D (not in the parametric space !!!)

## Parametric representation

- Curvatures of the surface  $\kappa_n = \frac{\varphi_2(d\Gamma^{uv}, d\Gamma^{uv})}{\varphi_1(d\Gamma^{uv}, d\Gamma^{uv})}$ 
  - The normal curvature  $\kappa_n$  at a point only depends on the local orientation of  $\Gamma$  on the surface. If we sweep through all the orientations, this curvature passes by a minimum  $\kappa_{min}$  then a maximum  $\kappa_{max}$  for two perpendicular directions.
  - These directions and the particular values of  $\kappa_n$  are respectively called **principal directions** and **principal curvatures** of the surface. They do not depend on the parametrization of the surface...
  - They are the solution of an eigenvalue problem.
  - The **Shape operator** (also called **Weingarten Map**) plays a key role here, see what follows.

## Parametric representation

- The shape operator  $W_p(w) = -\nabla_w \cdot N$      $w = w_u \cdot P_u + w_v \cdot P_v$ 
  - Here,  $N$  is the normal to the surface,  $p$  is a point on the surface,  $w$  is a tangent unit vector to the surface at  $p$ .  $\nabla_w$  is a directional derivative along  $w$  (in the surface) :  
 it is equal to  $w_u \frac{\partial}{\partial u} + w_v \frac{\partial}{\partial v}$ .  
 This operator gives back a vector in the tangent plane.



## Parametric representation

- The shape operator  $W_p(w) = -\nabla_w \cdot N$ 
  - If one uses an orthonormal basis, eigenvectors are easy to compute as the operator is **symmetric**...
  - Let's build such an orthonormal basis from the natural one :

$$P^u = \frac{\partial P}{\partial u} \quad P^v = \frac{\partial P}{\partial v} \quad N = N(u, v) = \frac{P^u \times P^v}{\|P^u \times P^v\|}$$

$$\longrightarrow t_1 = \frac{P^u}{\|P^u\|} \quad t_2 = N \times t_1 = \frac{P^v - (P^v \cdot t_1)t_1}{\|P^v - (P^v \cdot t_1)t_1\|}$$

## Parametric representation

- It is then easy to show that one can switch from one basis to the other using coefficients of the 1<sup>st</sup> fundamental form ...

$$t_1 = \frac{P^u}{\|P^u\|} = \frac{P^u}{\sqrt{e}} \quad t_2 = \frac{eP^v - fP^u}{\sqrt{e}\sqrt{eg - f^2}} = \frac{eP^v - fP^u}{\sqrt{e}\sqrt{h}}$$

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \frac{1}{\sqrt{e}\sqrt{h}} \begin{bmatrix} \sqrt{h} & 0 \\ -f & e \end{bmatrix} \begin{bmatrix} P^u \\ P^v \end{bmatrix} \quad h = eg - f^2 = \det(M_1)$$

- One can also switch the other way back, working on coefficients rather than on vectors e.g. for any vector  $\vec{w} = w_1 t_1 + w_2 t_2 = w_u P^u + w_v P^v$

One gets

$$\begin{bmatrix} w_u \\ w_v \end{bmatrix} = \frac{1}{\sqrt{e}\sqrt{h}} \begin{bmatrix} \sqrt{h} & -f \\ 0 & e \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \Phi \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

## Parametric representation

- Now back to the shape operator : we will start with a vector expressed in the orthonormal frame  $(w_1, w_2)$  , convert it to the parametric basis  $(w_u, w_v)$ , take the directional derivative  $w_u \frac{\partial}{\partial u} + w_v \frac{\partial}{\partial v}$  of the normal  $N$  and then project back to the orthonormal frame.

$$W_p(w) = - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} [n_u \quad n_v] \Phi \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= - \Phi^T \begin{bmatrix} P^u \\ P^v \end{bmatrix} [n_u \quad n_v] \Phi \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- Here,  $-\begin{bmatrix} P^u \\ P^v \end{bmatrix} [n_u \quad n_v]$  is precisely the matrix of the coefficients of the second fundamental form...



## Parametric representation

- In the orthonormal basis

$$\begin{aligned}
 W_p(w) &= \Phi^T \begin{bmatrix} l & m \\ m & n \end{bmatrix} \Phi \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{eh} \begin{bmatrix} \sqrt{h} & 0 \\ -f & e \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} \sqrt{h} & -f \\ 0 & e \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
 &= \frac{1}{eh} \begin{bmatrix} hl & (em - fl)\sqrt{h} \\ (em - fl)\sqrt{h} & f^2l + e^2n - 2efm \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
 \end{aligned}$$

- In the original basis

$$\begin{aligned}
 \tilde{W}_p(w) &= \Phi W_p(w) \Phi^{-1} \begin{bmatrix} w_u \\ w_v \end{bmatrix} = \Phi \left( \Phi^T \begin{bmatrix} l & m \\ m & n \end{bmatrix} \Phi \right) \Phi^{-1} \begin{bmatrix} w_u \\ w_v \end{bmatrix} \\
 &= \Phi \Phi^T \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} w_u \\ w_v \end{bmatrix} = \frac{1}{h} \begin{bmatrix} g & -f \\ -f & e \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} w_u \\ w_v \end{bmatrix} = \begin{bmatrix} e & f \\ f & g \end{bmatrix}^{-1} \cdot \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} w_u \\ w_v \end{bmatrix}
 \end{aligned}$$

(Weingarten Equation)

## Parametric representation

- Eigenvalues and eigenvectors of the shape operator
  - The shape operator tells the behavior of the normal for a given orientation (*i.e.* the derivative of the unit normal vector with respect to a small increment along that orientation on the surface)
  - One can work either in the original frame (non orthonormal):

$$(L - \kappa I) \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ with } L = M_1^{-1} M_2 = \tilde{W}_p(w) \quad \det(L - \kappa I) = 0$$

... or the orthonormal frame, for which the matrix is symmetric and the 2D eigenvectors are orthogonal

$$(W_p(w) - \kappa I) \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \det(W_p(w) - \kappa I) = 0$$

## Parametric representation

$$(M_2 - \kappa M_1) \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} l - \kappa e & m - \kappa f \\ m - \kappa f & n - \kappa g \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- The eigenvectors are obtained using any of the relations:

$$\begin{pmatrix} du \\ dv \end{pmatrix} \cdot \begin{pmatrix} l - \kappa e \\ m - \kappa f \end{pmatrix} = 0 \longrightarrow \begin{pmatrix} du \\ dv \end{pmatrix} = \alpha \cdot \begin{pmatrix} m - \kappa f \\ \kappa e - l \end{pmatrix}, \quad \alpha > 0: du^2 + dv^2 = 1$$

- Just use the other relation if the basis contains a null vector for one of the values of  $\kappa$  :

$$\begin{pmatrix} du \\ dv \end{pmatrix} \cdot \begin{pmatrix} m - \kappa f \\ l - \kappa g \end{pmatrix} = 0 \longrightarrow \begin{pmatrix} du \\ dv \end{pmatrix} = \alpha \cdot \begin{pmatrix} l - \kappa g \\ \kappa f - m \end{pmatrix}, \quad \alpha > 0: du^2 + dv^2 = 1$$

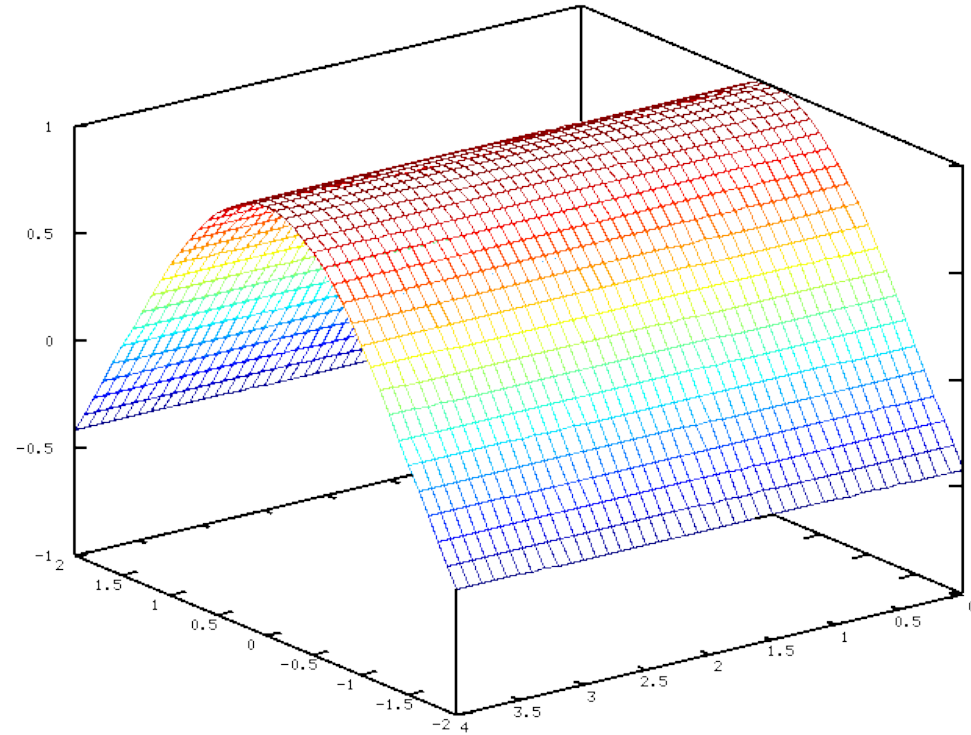
## Parametric representation

- Example

$$P(u, v) = \begin{pmatrix} u \\ v \\ \cos u \end{pmatrix}$$

$$P^u(u, v) = \begin{pmatrix} 1 \\ 0 \\ -\sin u \end{pmatrix}$$

$$P^v(u, v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



$$P^{uu}(u, v) = \begin{pmatrix} 0 \\ 0 \\ -\cos u \end{pmatrix}$$

$$P^{uv}(u, v) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$P^{vv}(u, v) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$N(u, v) = \frac{P^u \times P^v}{|P^u \times P^v|} = \begin{pmatrix} -\sin u \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{1 + \sin^2 u}}$$

## Parametric representation

$$(L - \kappa I) \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ with } L = M_1^{-1} M_2$$

$$M_1 = \begin{pmatrix} P^u \cdot P^u & P^u \cdot P^v \\ P^u \cdot P^v & P^v \cdot P^v \end{pmatrix} = \begin{pmatrix} 1 + \sin^2 u & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} N \cdot P^{uu} & N \cdot P^{uv} \\ N \cdot P^{uv} & N \cdot P^{vv} \end{pmatrix} = \begin{pmatrix} \frac{-\cos u}{\sqrt{1 + \sin^2 u}} & 0 \\ 0 & 0 \end{pmatrix}$$

## Parametric representation

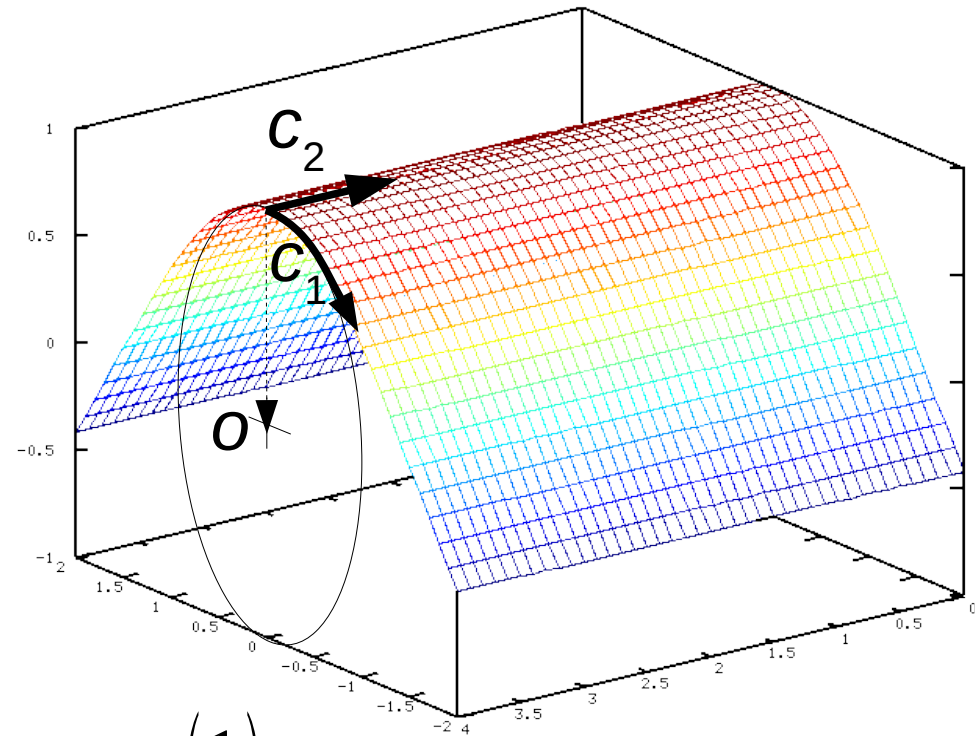
- Numerical application :  $u=0$  and  $v=0$

$$M_1 = \begin{pmatrix} 1 + \sin^2 u & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \frac{-\cos u}{\sqrt{1 + \sin^2 u}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L = M_1^{-1} M_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} \kappa_1 = -1 & c_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \kappa_2 = 0 & c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix} \rightarrow \begin{matrix} C_1^{3D} = (P_u, P_v) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ C_2^{3D} = (P_u, P_v) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$



## Parametric representation

### Umbilical points

- Points whose normal curvature  $\kappa_n$  is independent of the orientation of  $\Gamma$ .
  - Locally, the surface looks like a plane or a sphere.
- In this case, the following relations hold for the matrices  $M_1$  and  $M_2$  :

$$\frac{l}{e} = \frac{m}{f} = \frac{n}{g} \text{ if } f \neq 0$$

$$\frac{l}{e} = \frac{n}{g} \text{ if } f = 0 \text{ and } m = 0$$

$$\kappa_n = \frac{l}{e} = \kappa_{min} = \kappa_{max} = \text{const.}$$

Principal directions are **arbitrary**.

## Parametric representation

- Gaussian curvature and average curvature
  - Invariants (do not depend on the parametrization)

$$\kappa_T = \kappa_{min} \kappa_{max} = \det(L) = \frac{\det(M_2)}{\det(M_1)} = \frac{ln - m^2}{eg - f^2}$$

$$\kappa_M = \frac{\kappa_{min} + \kappa_{max}}{2} = \frac{1}{2} \text{trace}(L) = \frac{1}{2} \frac{en + gl - 2fm}{eg - f^2}$$

$$(L = M_1^{-1} M_2)$$



## Parametric representation

### Local nature of the surface

- $\kappa_T > 0$  the surface is locally an ellipsoid – elliptic point
- $\kappa_T < 0$  : hyperbolic paraboloid – saddle point – hyperbolic point
- $\kappa_T = 0$  and  $\kappa_M \neq 0$  : parabolic cylinder – parabolic point
- $\kappa_T = 0$  and  $\kappa_M = 0$  : no information – locally plane

## Christoffel symbols

- Complete developments. Christoffel symbols
  - Expression of the first derivatives along a curve...

$$P_u = \frac{\partial P(u, v)}{\partial u} \quad u' = \frac{du(t)}{dt}$$

$$P_v = \frac{\partial P(u, v)}{\partial v} \quad v' = \frac{dv(t)}{dt}$$

- Lets differentiate that expression again...

$$\Gamma'(t) = u' P_u + v' P_v$$

## Christoffel symbols

- $$(\Gamma'(t) = u' P_u + v' P_v)'$$

$$\begin{aligned} \Gamma''(t) &= u'' P_u + u'(u' P_{uu} + v' P_{uv}) + v'' P_v + v'(u' P_{vu} + v' P_{vv}) \\ &= u'' P_u + v'' P_v + u'^2 P_{uu} + u'v' P_{uv} + u'v' P_{vu} + v'^2 P_{vv} \end{aligned}$$

- The terms  $u'' P_u + v'' P_v$  are in the tangent plane
- The terms  $P_{ij}, i, j = \{u, v\}$  can be expressed as linear combinations of tangential and normal components :

$$P_{ij} = C_{ij}^u P_u + C_{ij}^v P_v + L_{ij} N \leftarrow \text{Gauss formula}$$

The  $C_{ij}^k$  are the Christoffel symbols (often written  $\Gamma_{ij}^k$  in textbooks ) and the  $L_{ij}$  are the coefficients of the second fundamental form.

## Christoffel symbols

- Using Einstein's notation (summation over repeated indices ):  $P_{ij} = C_{ij}^k P_k + L_{ij} N$ ,  $i, j, k = u, v$ 
  - How to compute  $C_{ij}^k$  and  $L_{ij}$ ?

→ Multiply by  $P_l$ ,  $l = u, v$

$$P_{ij} \cdot P_l = C_{ij}^k (P_k \cdot P_l) + L_{ij} (N \cdot P_l)$$

Coefficients of the 1<sup>st</sup> fundamental form =  $g_{kl}$  = metric tensor

$$P_{ij} \cdot P_l \cdot (g_{kl})^{-1} = C_{ij}^k g_{kl} \cdot g_{kl}^{-1}$$

Conventional notation of the inverse of the metric tensor

$$C_{ij}^k = P_{ij} \cdot P_l \cdot (g_{kl})^{-1} \longrightarrow C_{ij}^k = P_{ij} \cdot P_l \cdot g^{lk}$$

## Christoffel symbols

$$P_{ij} = C_{ij}^k P_k + L_{ij} N$$

- Now, multiply by  $N$

$$P_{ij} \cdot N = C_{ij}^k \underbrace{P_k \cdot N}_0 + L_{ij} \underbrace{N \cdot N}_1$$

$$L_{ij} = P_{ij} \cdot N$$

- One recovers the formula seen before:

$$L_{ij} \equiv M_2 \quad M_2 = \begin{pmatrix} N \cdot P^{uu} & N \cdot P^{uv} \\ N \cdot P^{uv} & N \cdot P^{vv} \end{pmatrix}$$

## Christoffel symbols

- Use of Christoffel symbols and the second fundamental form

- Let us return to the expression

$$\Gamma''(t) = u'' P_u + v'' P_v + u'^2 P_{uu} + u'v' P_{uv} + u'v' P_{vu} + v'^2 P_{vv}$$

- Now replace  $u, v$  by  $u^i, i=1,2$  and use the Einstein notation again :

$$\Gamma''(t) = (u^i)'' P_i + (u^i)' \cdot (u^j)' P_{ij}$$

- Now substitute  $P_{ij} = C_{ij}^k P_k + L_{ij} N$

$$\Gamma''(t) = \left( (u^k)'' + C_{ij}^k (u^i)' \cdot (u^j)' \right) P_k + (u^i)' \cdot (u^j)' L_{ij} N$$

Total tangential component

Normal component

## Christoffel symbols

- Used to express curvatures (or second derivatives) in the 3D euclidean space from curvatures (or second derivatives) expressed in the parametric space.

- Normal curvature: based on the normal component  $(u^i)' \cdot (u^j)' L_{ij} N$

If we have an arc-length parametrization :

$$\kappa_n = \Gamma'' \cdot N = L_{ij} (u^i)' (u^j)' \quad ( = \varphi_2(d\Gamma^{uv}, d\Gamma^{uv}))$$

if not : need to adjust

$$\kappa_n = \frac{L_{ij} (u^i)' (u^j)'}{ds^2} = \frac{L_{ij} (u^i)' (u^j)'}{g_{kl} (u^k)' (u^l)'}$$

- One recovers the formula :  $\kappa_N = \frac{\varphi_2(d\Gamma^{uv}, d\Gamma^{uv})}{\varphi_1(d\Gamma^{uv}, d\Gamma^{uv})}$

## Christoffel symbols

- Geodesic curvature : based on the tangential components

$$\left( (u^k)'' + C_{ij}^k (u^i)' \cdot (u^j)' \right) P_k$$

- If the curve is parametrized by arc-length :

$$\Gamma' \cdot \Gamma' = 1 \quad \Gamma' \cdot \Gamma'' = 0$$

$$\Gamma'' \cdot \Gamma' = 0 = \left( (u^k)'' + C_{ij}^k (u^i)' \cdot (u^j)' \right) P_k \cdot \Gamma' + (u^i)' \cdot (u^j)' L_{ij} N \cdot \Gamma' \quad \begin{matrix} =0 \\ \uparrow \end{matrix}$$

- Therefore, the tangential component  $\Gamma_t''$  of  $\Gamma''$  is orthogonal to  $\Gamma'$  and to  $N$  as well. There is a proportionality factor between  $\Gamma_t''$  and  $N \times \Gamma'$ . This factor is the geodesic curvature



## Christoffel symbols

- $\Gamma_t'' = \kappa_g (N \times \Gamma')$       $\Gamma_t'' \cdot (N \times \Gamma') = \kappa_g (N \times \Gamma') \cdot (N \times \Gamma')$   
 $\kappa_g = (N \times \Gamma') \cdot \Gamma_t'' = (N \times \Gamma') \cdot \Gamma''$

Therefore,  $\kappa_g = [N, \Gamma', \Gamma'']$ , valid only if the parametrization of  $\Gamma$  is normal (arc-length)

- If not, need to adjust ...

$$\kappa_g = \frac{[N, \Gamma', \Gamma'']}{ds^3} = \frac{[N, \Gamma', \Gamma'']}{\left(g_{kl} (u^k)' (u^l)'\right)^{3/2}}$$

- Some developments\* leads to the following identity :

$$\kappa_g = \frac{\sqrt{\det(M_1)} [-C_{11}^2 u'^3 + C_{22}^1 v'^3 - (2C_{12}^2 - C_{11}^1) u'^2 v' + (2C_{12}^1 - C_{22}^2) u' v'^2 + u'' v' - v'' u']}{\left((u' \quad v') M_1 \begin{pmatrix} u' \\ v' \end{pmatrix}\right)^{3/2}}$$

Again, this expression makes only use of derivatives in the parametric space...

\* see e.g. "Modern Differential Geometry of Curves and Surfaces with Mathematica", 2nd ed. Boca Raton, FL: CRC Press, pp. 501-518, 1997.

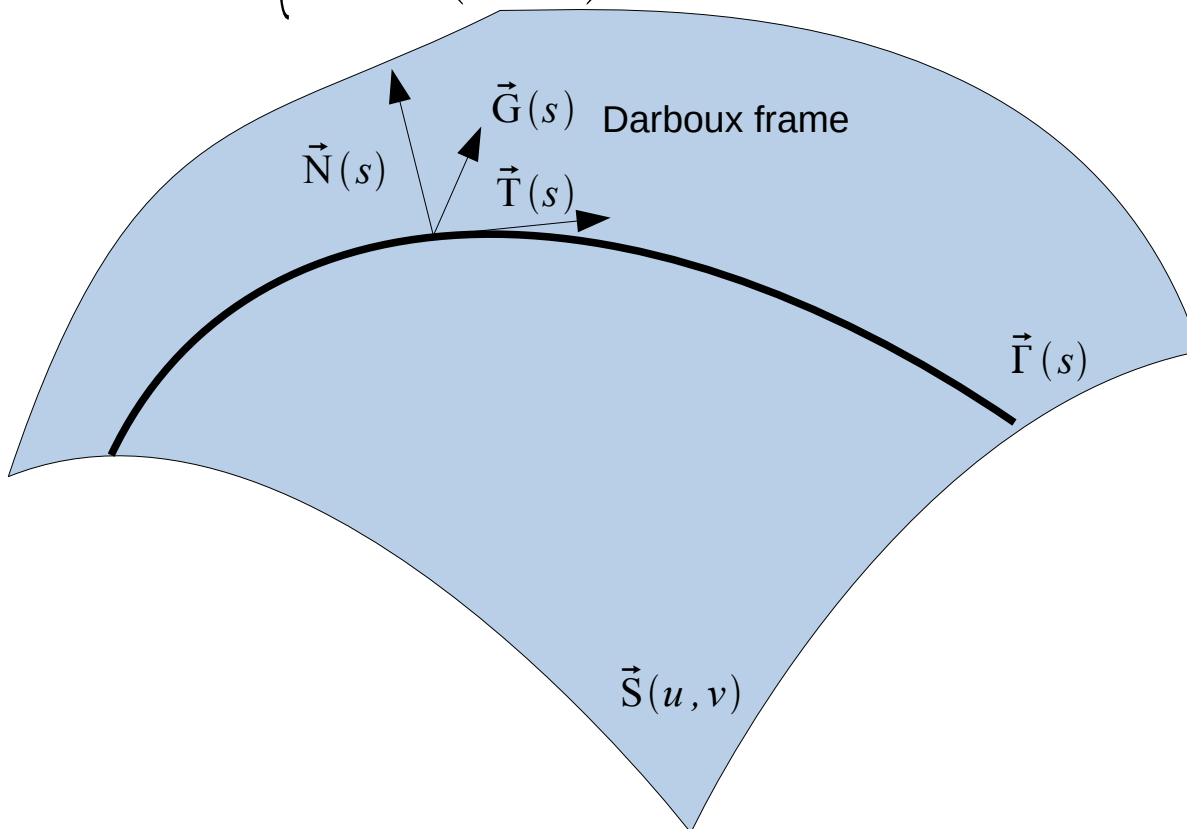
## Christoffel symbols

- Equations of a geodesic
  - Need to set that the geodesic curvature is equal to zero.
  - One obtains a set of differential equations
  - May be used to compute an approximation of the geodesic since it is usually impossible to solve that coupled system of ODEs algebraically.

## Parametric representation

### Darboux frame and equations

$$\vec{P}(u, v): \begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases} \quad \vec{\Gamma}^{uv}(s): \begin{cases} u = u(s) \\ v = v(s) \end{cases} \quad \vec{\Gamma}(s): \begin{cases} x(s) = f(u(s), v(s)) \\ y(s) = g(u(s), v(s)) \\ z(s) = h(u(s), v(s)) \end{cases}$$



$$\vec{T}(s) = \left\{ \frac{\partial x(s)}{\partial s}, \frac{\partial y(s)}{\partial s}, \frac{\partial z(s)}{\partial s} \right\}$$

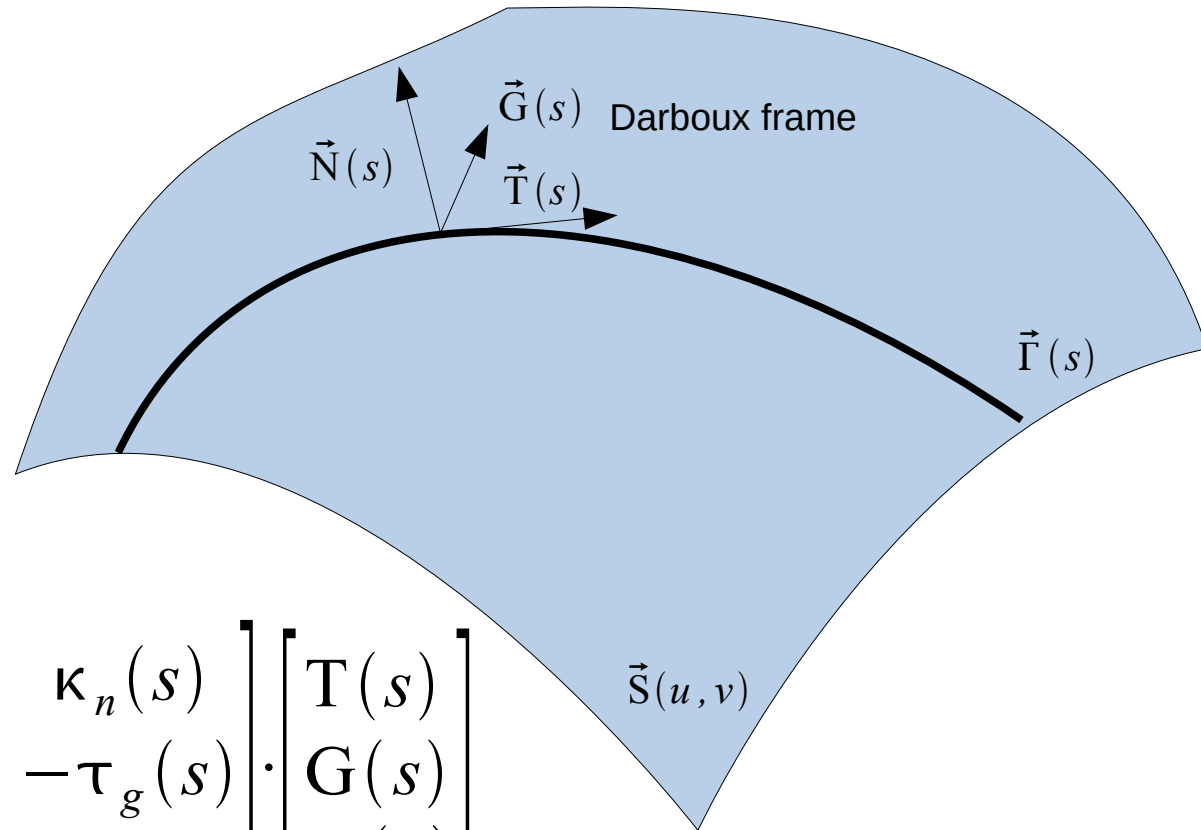
$$\vec{N}(s) = \vec{N}(u(s), v(s))$$

$$\vec{G}(s) = \vec{N} \times \vec{T}$$

$s$  is the curvilinear abscissa

# Parametric representation

## Darboux frame and equations



$$\begin{bmatrix} \frac{d \mathbf{T}(s)}{ds} \\ \frac{d \mathbf{G}(s)}{ds} \\ \frac{d \mathbf{N}(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ -\kappa_g(s) & 0 & -\tau_g(s) \\ -\kappa_n(s) & \tau_g(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{N}(s) \end{bmatrix}$$

Geodesic torsion (or relative torsion)

## Parametric representation

- Geodesic torsion

is identically zero iff the curve is along a direction of principal curvature (line of curvature)

- Geodesic curvature

is identically zero iff the curve is a geodesic

## Parametric representation

- In an euclidean space:
  - The shortest line between two points has simple curvature  $\kappa$  equal to zero (= straight line)
  - The minimal surface carried by a curve is of average curvature  $\kappa_M$  equal to 0 everywhere
- In a non euclidean space (on a surface):
  - The (or one of the) shortest line between two points on the surface has a geodesic curvature  $\kappa_g$  equal to zero (it's a geodesic)

## Parametric representation

- A good reference on « shape interrogation »
  - Shape interrogation is the ability to get information from geometric models

N. M. Patrikalakis and T. Maekawa, *Shape Interrogation for Computer Aided Design and Manufacturing*, Springer Verlag, February 2002.  
ISBN 3-540-42454-7

(Available as a PDF electronic copy at the library)

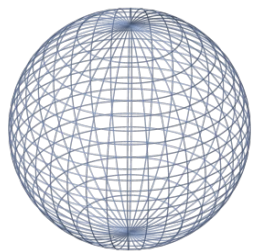
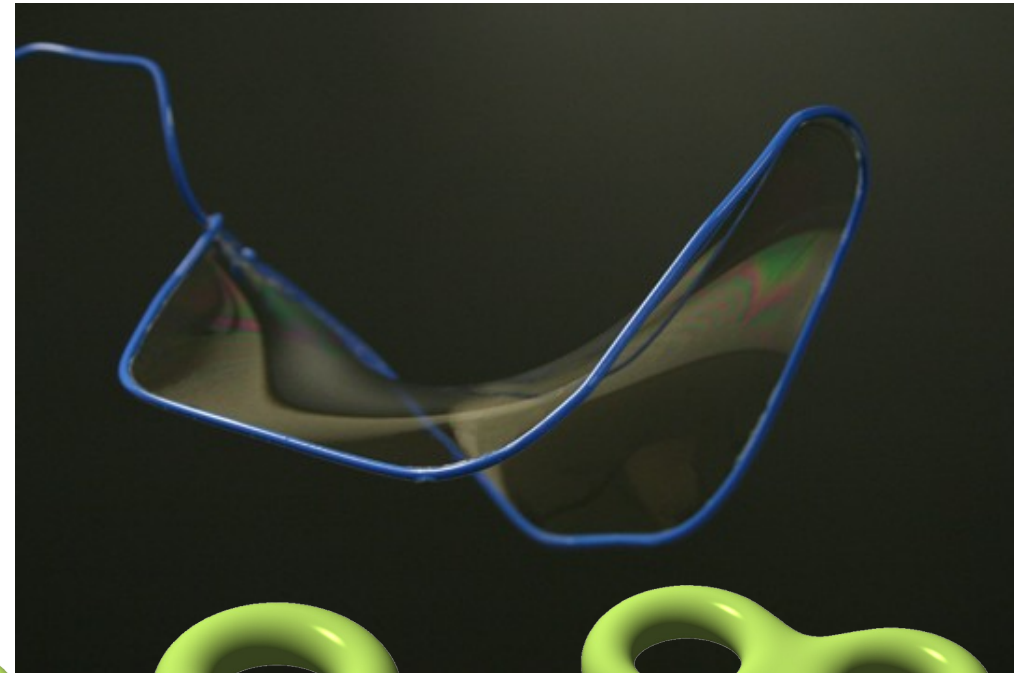
## Parametric representation

### Some results of differential geometry

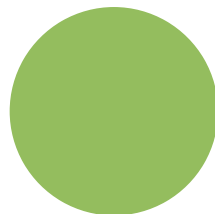
#### Gauss-Bonnet's theorem

$$\int_S \kappa_T dA + \int_{\partial S} \kappa_g ds = 2\pi \chi(S)$$

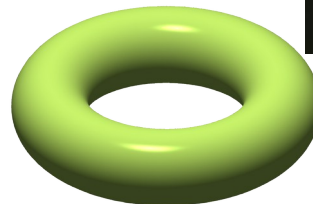
$\chi(S)$  is Euler's characteristic of surface  $S$



$\chi = 2$



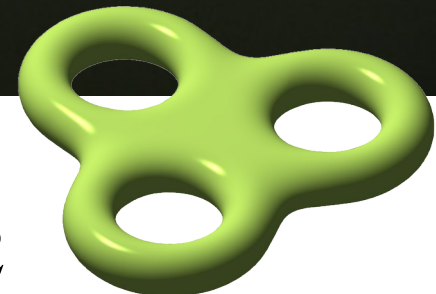
$\chi = 1$



$\chi = 0$



$\chi = -2$



$\chi = -4$



## Parametric representation

### Application of Gauss-Bonnet's theorem

- For a polyhedron, we have (Euler's formula) :

$$\chi(S) = N - E + F$$

We want to approximate a sphere with hexagons ( $H$ ) and pentagons ( $P$ ).

- Each pentagon (hexagon) has 5 (6) nodes, shared by 3 faces
- Each pentagon (hexagon) has 5 (6) edges, shared by 2 faces
- So the total number of nodes  $N$  is  $(5P+6H)/3$
- Total number of edges  $E$  is  $(5P+6H)/2$ , total number of faces  $F$  is  $P+H$

$$\chi(S) = N - E + F = (5P + 6H)/3 - (5P + 6H)/2 + P + H = P/6$$

- The sphere is compact and its gaussian curvature is  $1/R^2$

$$\int_S \kappa_T dA + \int_{\partial S} \kappa_G ds = \frac{1}{R^2} \int_S dA = 4\pi = 2\pi \chi(S)$$

so  $P=12$  ... 12 pentagons are necessary and as many hexagons as we want to make soccer ball.

## Parametric representation

